

CRITICAL POINTS AND SUPERSYMMETRIC VACUA, II: ASYMPTOTICS AND EXTREMAL METRICS

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Abstract

Motivated by the vacuum selection problem of string/M theory, we study a new geometric invariant of a positive Hermitian line bundle $(L, h) \rightarrow M$ over a compact Kähler manifold: the expected distribution of critical points of a Gaussian random holomorphic section $s \in H^0(M, L)$ with respect to the Chern connection ∇_h . It is a measure on M whose total mass is the average number $\mathcal{N}_h^{\text{crit}}$ of critical points of a random holomorphic section. We are interested in the metric dependence of $\mathcal{N}_h^{\text{crit}}$, especially metrics h which minimize $\mathcal{N}_h^{\text{crit}}$. We concentrate on the asymptotic minimization problem for the sequence of tensor powers $(L^N, h^N) \rightarrow M$ of the line bundle and their critical point densities $\mathcal{K}_{N,h}^{\text{crit}}(z)$. We prove that $\mathcal{K}_{N,h}^{\text{crit}}(z)$ has a complete asymptotic expansion in N whose coefficients are curvature invariants of h . The first two terms in the expansion of $\mathcal{N}_{N,h}^{\text{crit}}$ are topological invariants of (L, M) . The third term is a topological invariant plus a constant $\beta_2(m)$ (depending only on the dimension m of M) times the Calabi functional $\int_M \rho^2 d\text{Vol}_h$, where ρ is the scalar curvature of the Kähler metric $\omega_h := \frac{i}{2}\Theta_h$. We give an integral formula for $\beta_2(m)$ and show, by a computer assisted calculation, that $\beta_2(m) > 0$ for $m \leq 5$, hence that $\mathcal{N}_{N,h}^{\text{crit}}$ is asymptotically minimized by the Calabi extremal metric (when one exists). We conjecture that $\beta_2(m) > 0$ in all dimensions, i.e., the Calabi extremal metric is always the asymptotic minimizer.

1. Introduction

This paper is the second in a series of articles [11, 12] on the statistics of vacua in string/M theory and associated effective supergravity theories. Mathematically, vacua are critical points $\nabla s(z) = 0$ of a holomorphic section $s \in H^0(M, L)$ of a line bundle $L \rightarrow M$ over a complex manifold relative to a connection ∇ , which we always choose to be the Chern connection ∇_h of a Hermitian metric h on L . Equivalently, they

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are critical points of the norm $|s(z)|_h$ of s relative to h . Our motivation to study critical points of holomorphic sections in this metric sense comes initially from physics, where critical points of this kind model extremal black holes in addition to string/M vacua (cf. [13, 20, 22]). But we also find the statistics of critical points to have an independent geometric interest. A basic statistical quantity, the average number of critical points of a random holomorphic section, defines a new geometric invariant of a positive Hermitian holomorphic line bundle, and in this paper we show that its asymptotic minima are given by Calabi extremal metrics.

The physical setting for vacua of string/M theory (and for extremal black holes) is a holomorphic line bundle over the moduli space of complex structures on a polarized Calabi-Yau manifold. Supersymmetric vacua are critical points of a holomorphic section (known as a superpotential) relative to the Weil-Petersson metric. The program of studying statistics of critical points of Gaussian random holomorphic sections was proposed by the first author [1, 8, 10] as a means of dealing with the large number of string/M vacua. There exists at this time no reasonable selection principle to decide which superpotential nor which of its critical points gives the vacuum state that correctly describes our universe in string/M theory, so it makes sense to study the statistics of vacua of random superpotentials.

In our first paper [11], we gave explicit formulas for the expected distribution of critical points of sections of general holomorphic line bundles over any complex manifold, including those which arise in physics. The formulas, recalled in §2 (cf. Theorem 2.1), involve complicated complex symmetric matrix integrals, and it is difficult to see how the expected distribution of critical points depends on the metric h . The purpose of this article is therefore to study a purely geometric simplification of the physical problem where (L, h) is a positive Hermitian line bundle over a compact Kähler manifold M and where the Gaussian measure on $H^0(M, L)$ is derived from the inner product induced by h . Our aim is to understand the metric dependence of the statistics of the random critical point set

$$(1) \quad \text{Crit}(s, h) = \{z : \nabla_h(s) = 0\}$$

of a Gaussian random section of $H^0(M, L)$. We note that $\text{Crit}(s, h) \cup Z_s = \{z : d|s(z)|_h^2 = 0\}$ where Z_s is the zero set of s .

From the probabilistic viewpoint, the critical points of random holomorphic sections relative to the Chern connection ∇_h of a fixed Hermitian metric on L determine a *point process* on M , that is, a measure on the configuration space of finite subsets of M , which gives the probability density of a finite subset being the critical point set of a holomorphic section. The process is determined by its n -point correlation functions

$\mathbf{K}_n^{\text{crit}}(z_1, \dots, z_n)$, which give the probability density of critical points occurring simultaneously at the points $z_1, \dots, z_n \in M$.

In this article, we focus on the 1-point correlation, namely the expected distribution of critical points

$$(2) \quad \mathbf{K}_h^{\text{crit}} = \int \left[\sum_{z \in \text{Crit}(s, h)} \delta_z \right] d\gamma_h(s)$$

where δ_z is the Dirac point mass at z , and where γ_h is the Gaussian measure probability γ_h on $H^0(M, L)$ induced by h and the Kähler form $\omega_h := \frac{i}{2}\Theta_h$ (see §2). We showed in [11] that $\mathbf{K}_h^{\text{crit}}$ is a smooth form on M if L is sufficiently positive. In particular, we are interested in the expected (average) number of critical points

$$(3) \quad \mathcal{N}_h^{\text{crit}} = \int_M \mathbf{K}_h^{\text{crit}} = \int \#\text{Crit}(s, h) d\gamma_h(s)$$

of a random section, a purely geometric invariant of a positive Hermitian holomorphic line bundle $(L, h) \rightarrow M$. The methods of this paper also give results on general n -point correlations between critical points and their scaling asymptotics in the sense of [3], but we do not carry out the analysis here.

Since it is a crucial point, let us explain why the number $\#\text{Crit}(s, h)$ is a (non-constant) random variable on $H^0(M, L)$, unlike the number of zeros of m independent sections which is a topological invariant of L . As indicated above, connection critical points are the same as critical points of $|s(z)|_h^2$ for which $s(z) \neq 0$, or equivalently as critical points of $\log |s(z)|_h$ (see [11] for the simple proof). Hence, there are critical points of each Morse index $\geq m$ (see [5, 11]), and only the alternating sum of the number of critical points of each index is a topological invariant. Another way to understand the metric dependence of the number of critical points is to write the covariant derivative in a local frame e_L as

$$(4) \quad \nabla_{z_j} s = \left(\frac{\partial f}{\partial z_j} - f \frac{\partial K}{\partial z_j} \right) e_L = e^K \frac{\partial}{\partial z_j} (e^{-K} f) e_L, \quad \nabla_{\bar{z}_j} s = \frac{\partial f}{\partial \bar{z}_j} e_L,$$

where we locally express a section as $s = f e_L$, and $K = -\log \|e_L\|_h^2$. Hence, the critical point equation

$$(5) \quad \left(\frac{\partial f}{\partial z_j} - f \frac{\partial K}{\partial z_j} \right) = 0$$

in the local frame fails to be holomorphic when the connection form is only smooth.

Although $\text{Crit}(s, h)$ and $\#\text{Crit}(s, h)$ depend on h , it is not clear at the outset whether $\mathcal{N}_h^{\text{crit}}$ is a topological invariant or to what degree it depends on the metric h . To investigate the metric dependence of $\mathbf{K}_h^{\text{crit}}$ and $\mathcal{N}_h^{\text{crit}}$ we consider their asymptotic behavior as we take powers L^N of L . As in [3, 21], it is natural to expect that the density and number of

critical points will have asymptotic expansions which reveal their metric dependence.

We therefore let $\mathbf{K}_{N,h}^{\text{crit}}(z)$ denote the expected distribution of critical points of random holomorphic sections $s_N \in H^0(M, L^N)$ with respect to the Chern connection and Hermitian Gaussian measure induced by h , as given by (15)–(16) in §2. We also let

$$(6) \quad \mathcal{N}_{N,h}^{\text{crit}} = \int_M \mathbf{K}_{N,h}^{\text{crit}}(z)$$

denote the expected number of critical points.

We let $\Theta_h = \partial\bar{\partial}K$ denote the curvature form of (L, h) . Our first result is a complete asymptotic expansion for the expected distribution of critical points for powers $L^N \rightarrow M$ in terms of curvature invariants of the Kähler metric $\omega_h = \frac{i}{2}\Theta_h$:

Theorem 1.1. *For any positive Hermitian line bundle $(L, h) \rightarrow (M, \omega_h)$ over any compact Kähler manifold with $\omega_h = \frac{i}{2}\Theta_h$, the expected critical point distribution of holomorphic sections of L^N relative to the Hermitian Gaussian measure induced by h and ω_h has an asymptotic expansion of the form*

$$N^{-m} \mathbf{K}_{N,h}^{\text{crit}}(z) \sim \{b_0 + b_1(z)N^{-1} + b_2(z)N^{-2} + \dots\} \frac{\omega_h^m}{m!},$$

where the coefficients $b_j = b_j(m)$ are curvature invariants of order j of ω_h . In particular, b_0 is the universal constant

$$(7) \quad b_0 = \pi^{-\binom{m+2}{2}} \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} |\det(2HH^* - |x|^2 I)| e^{-\langle(H,x), (H,x)\rangle} dH dx,$$

$b_1 = \beta_1 \rho$, where ρ is the scalar curvature of ω_h and β_1 is a universal constant, and b_2 is a quadratic curvature polynomial.

Here, $\text{Sym}(m, \mathbb{C})$ is the space of $m \times m$ complex symmetric matrices. It follows that the density of critical points is asymptotically uniform relative to the curvature volume form with a universal asymptotic density $b_0(m)$. The values of the constant b_0 for low dimensions are:

$$b_0(1) = \frac{5}{3\pi}, \quad b_0(2) = \frac{59 \cdot 2}{3^3 \pi^2}, \quad b_0(3) = \frac{637 \cdot 3!}{3^5 \pi^3},$$

$$b_0(4) = \frac{6571 \cdot 4!}{3^7 \pi^4}, \quad b_0(5) = \frac{65941 \cdot 5!}{3^9 \pi^5}, \quad b_0(6) = \frac{649659 \cdot 6!}{3^{11} \pi^6}.$$

(See Appendix 1.)

With only minor changes in the proofs, our methods give refinements of the asymptotic results which take the Morse indices of the critical points into account. By the Morse index q of a critical point, we mean its Morse index as a critical point of $\log \|s\|_{h,N}$; it is well known that $m \leq q \leq 2m$ for positive line bundles [5]. Thus we let $\mathbf{K}_{N,q,h}^{\text{crit}}(z) = \mathbf{K}_{N,q}^{\text{crit}}(z)$

denote the expected distribution of critical points of $\log \|s_N\|_{h^N}$ of Morse index q , and we let $\mathcal{N}_{N,q,h}^{\text{crit}}$ denote the expected number of these critical points. Thus we have

$$(8) \quad \mathbf{K}_{N,h}^{\text{crit}}(z) = \sum_{q=m}^{2m} \mathbf{K}_{N,q,h}^{\text{crit}}(z), \quad \mathcal{N}_{N,h}^{\text{crit}} = \sum_{q=m}^{2m} \mathcal{N}_{N,q,h}^{\text{crit}}.$$

We obtain a similar asymptotic expansion for the distribution of critical points of given Morse index:

Theorem 1.2. *Let $M, L, h, \omega_h, \mathbf{K}_{N,q,h}^{\text{crit}}$ be as above. Then the expected density of critical points of Morse index q of random sections in $H^0(M, L^N)$ is given by*

$$N^{-m} \mathbf{K}_{N,q,h}^{\text{crit}}(z) \sim \{b_{0q} + b_{1q}(z)N^{-1} + b_{2q}(z)N^{-2} + \dots\} \frac{\omega_h^m}{m!},$$

$m \leq q \leq 2m$, where the $b_{jq} = b_{jq}(m)$ are curvature invariants of order j of ω_h . In particular, b_{0q} is given by the integral in (7) with the domain of integration $\text{Sym}(m, \mathbb{C}) \times \mathbb{C}$ replaced by

$$(9) \quad \mathbf{S}_{m,k} := \{(H, x) \in \text{Sym}(m, \mathbb{C}) \times \mathbb{C} : \text{index}(2HH^* - |x|^2I) = k\},$$

with $k = q - m$.

The coefficient integrals are in general quite complicated to evaluate. In Appendix 1, we provide a table of numerical values of $b_{0q}(m)$ (for $m \leq 6$), which we computed using Maple. In dimension one, we have the following formula for the first three terms of the expansion:

Theorem 1.3. *Let (L, h) be a positive line bundle on a compact complex curve C of genus g . Then the expected number of saddle points and of local maxima, respectively, of random holomorphic sections of L^N are given by:*

$$\begin{aligned} \mathcal{N}_{N,1,h}^{\text{crit}} &= \frac{4}{3} c_1(L) N + \frac{8}{9} (2g - 2) + \left(\frac{1}{27\pi} \int_C \rho^2 \omega_h \right) N^{-1} + O(N^{-2}), \\ \mathcal{N}_{N,2,h}^{\text{crit}} &= \frac{1}{3} c_1(L) N - \frac{1}{9} (2g - 2) + \left(\frac{1}{27\pi} \int_C \rho^2 \omega_h \right) N^{-1} + O(N^{-2}), \end{aligned}$$

where $\omega_h = \frac{i}{2} \Theta_h$, and ρ is the Gaussian curvature of the metric ω_h .

Hence, the expected total number of critical points in dimension 1 is

$$\mathcal{N}_{N,h}^{\text{crit}} = \frac{5}{3} c_1(L) N + \frac{7}{9} (2g - 2) + \left(\frac{2}{27\pi} \int_C \rho^2 \omega_h \right) N^{-1} + O(N^{-2}).$$

In dimension 1, the coefficients in Theorem 1.2 are:

$$\begin{aligned} b_{01}(1) &= \frac{4}{3\pi}, \quad b_{02}(1) = \frac{1}{3\pi}, \quad b_{11}(1) = -\frac{8}{9\pi} \rho, \quad b_{12}(1) = \frac{1}{9\pi} \rho, \\ b_{21}(1) &= b_{22}(1) = \frac{1}{27\pi} \rho^2. \end{aligned}$$

In §4, we study the case where $M = \mathbb{C}\mathbb{P}^m$ and (L, h) is the hyperplane section bundle $\mathcal{O}(1)$ with the Fubini-Study metric. We show that in this case the expected number $\mathcal{N}_{N,q,h}^{\text{crit}}$ of critical points of Morse index q of sections in $H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(N))$ is a rational function of N , and we provide an exact formula (Proposition 4.1) for this expected number. In Appendix 1, we use Proposition 4.1 and a computer assisted computation to give explicit formulas for $\mathcal{N}_{N,q,h}^{\text{crit}}$ in dimensions ≤ 3 . For example, in dimension 1, the expected total number of critical points is

$$\mathcal{N}_{N,h}^{\text{crit}}(\mathbb{C}\mathbb{P}^1) = \frac{5N^2 - 8N + 4}{3N - 2},$$

as we computed in [11]; in dimension 2, the expected total number is

$$\mathcal{N}_{N,h}^{\text{crit}}(\mathbb{C}\mathbb{P}^2) = \frac{59N^5 - 231N^4 + 375N^3 - 310N^2 + 132N - 24}{(3N - 2)^3}.$$

Thus we gain a quantitative sense of how many more critical points there are in the metric sense in comparison with the classical critical point equation $\partial f = 0$. In the case of $\mathcal{O}(N) \rightarrow \mathbb{C}\mathbb{P}^1$, whose sections are polynomials of degree N , we may view this classical critical point equation as a connection critical point equation by viewing the derivative $\frac{\partial}{\partial z}$ as a flat meromorphic connection with pole at ∞ . Alternately, it is the Chern connection of a singular Hermitian metric. The critical point equation being purely holomorphic, the number of critical points of a generic section is a constant $N - 1$. All critical points relative to this connection are saddle points. By comparison, sections $s_N \in H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(N))$ have, on average, an additional $\sim \frac{N}{3}$ local maxima and $\sim \frac{N}{3}$ additional saddles relative to the Fubini-Study connection. The study of critical points relative to meromorphic connections (known as Minkowski vacua) is simpler than that relative to Chern connections and will be explored further in a subsequent work.

As a corollary of Theorem 1.2, we find the rather surprising fact that the asymptotics of the expected number of critical points is a topological invariant of the bundle $L \rightarrow M$ to two orders in N .

Corollary 1.4. *Let $(L, h) \rightarrow (M, \omega_h)$ be a positive holomorphic line bundle on a compact Kähler manifold, with $\omega_h = \frac{i}{2}\Theta_h$. Then the expected number of critical points of Morse index q ($m \leq q \leq 2m$) of random sections in $H^0(M, L^N)$ has the asymptotic expansion*

$$\begin{aligned} \mathcal{N}_{N,q,h}^{\text{crit}} \sim & \left[\frac{\pi^m b_{0q}}{m!} c_1(L)^m \right] N^m + \left[\frac{\pi^m \beta_{1q}}{(m-1)!} c_1(M) \cdot c_1(L)^{m-1} \right] N^{m-1} \\ & + \left[\beta_{2q} \int_M \rho^2 d\text{Vol}_h + \beta'_{2q} c_1(M)^2 \cdot c_1(L)^{m-2} \right. \\ & \left. + \beta''_{2q} c_2(M) \cdot c_1(L)^{m-2} \right] N^{m-2} + \dots, \end{aligned}$$

where $b_{0q}, \beta_{1q}, \beta_{2q}, \beta'_{2q}, \beta''_{2q}$ are universal constants depending only on the dimension m .

We recall that the alternating sum of the $\mathcal{N}_{N,q,h}^{\text{crit}}$ is topological:

$$\sum (-1)^{m+q} \mathcal{N}_{N,q,h}^{\text{crit}} = c_n(L^N \otimes T^{*1,0}) = c_1(L)^m N^m + \dots,$$

and hence by Corollary 1.4,

$$(10) \quad \sum_{q=m}^{2m} (-1)^{m+q} b_{0q}(m) = \frac{m!}{\pi^m}.$$

Since each b_{0q} is strictly positive and their sum equals b_0 , it follows from (10) that

$$(11) \quad b_0(m) > \frac{m!}{\pi^m}.$$

Corollary 1.4 shows that the metric dependence of $\mathcal{N}^{\text{crit}}(h^N)$ is a ‘lower-order’ effect, and hence renews the question to what degree $\mathcal{N}_{q,h}^{\text{crit}}$, or at least the asymptotic expansion of $\mathcal{N}_{N,q,h}^{\text{crit}}$, is a topological invariant. We see from Corollary 1.4 that the expansion is not topological provided that the constant $\beta_{2q} = \beta_{2q}(m)$ does not vanish. Computations in dimensions ≤ 5 show that β_{2q} is positive in these dimensions (see Corollary 1.7), and we conjecture that $\beta_{2q}(m) > 0$ for all m . As we now explain, this conjecture is suggested by a connection between extremals of $\mathcal{N}_{N,h}^{\text{crit}}$ and Calabi extremal metrics [6, 7, 9, 24].

This connection involves a notion of asymptotic minimality of $\mathcal{N}_{N,h}^{\text{crit}}$. To introduce it, we revisit the motivating problem of determining how $\mathcal{N}_h^{\text{crit}}$ varies as h varies over Hermitian metrics on L . One could consider all Hermitian metrics on L , but we focus on the smaller class of positively curved Hermitian metrics,

$$P(M, L) = \left\{ h : \frac{i}{2} \Theta_h \text{ is a positive } (1, 1)\text{-form} \right\}.$$

If we fix one such metric $h_0 = e^{-K_0}$, the others may be expressed as $h_\varphi := e^\varphi h_0$ with $\varphi \in C^\infty(M)$. It is reasonable to conjecture that $\mathcal{N}_{h_\varphi}^{\text{crit}}$ is unbounded as h_φ varies over $P(M, L)$, since by (5) the number of critical points of a section should be ‘large’ if the ‘degree’ of the connection form $-\partial K$ is ‘large’. Here, $K = -\log h = K_0 - \varphi$ in a local frame. On the other hand, $\mathcal{N}_h^{\text{crit}}$ is bounded below by $|c_m(L \otimes T^{*1,0})|$, and it is plausible that it has a minimum. It would be interesting to determine this minimal metric (assuming one exists and is unique), but it is difficult to solve the critical point equation $\delta \mathcal{N}_h^{\text{crit}} = 0$.

We therefore consider the simpler asymptotic problem. Since the first two leading coefficients in the expansion of $\mathcal{N}_{N,h}^{\text{crit}}$ are topological, we try

to find metrics for which the first non-topological term is critical. The first non-topological term is a multiple of the Calabi functional

$$\int_M \rho_h^2 dV_h,$$

where ρ_h is the scalar curvature of the Kähler metric $\omega_h = \frac{i}{2}\Theta_h$, and $dV_h = \frac{1}{m!}\omega_h^m$. Thus the problem of finding metrics which are critical for the metric invariant $\mathcal{N}_{N,h}^{\text{crit}}$ is closely related to the problem of finding critical points of Calabi’s functional. In keeping with our intuition that $\mathcal{N}_{N,h}^{\text{crit}}$ should have a minimizer, we note that critical points of Calabi’s functional are necessarily minima (cf. [6, 7, 17]).

Existence of critical metrics is one of the fundamental problems in complex geometry, and we refer to [9, 24, 26] for background. It was suggested by S.-T. Yau [25, 27] that existence of a canonical metric should be related to the stability of M . One class of canonical metrics are Hermitian metrics h for which Θ_h is a Kähler metric of constant scalar curvature, i.e. for which ρ_h is constant. By a theorem due to S. Donaldson [9, Cor. 5], there exists at most one Kähler metric of constant scalar curvature in the cohomology class of $\pi c_1(L)$. Hence if there exists such a metric of constant scalar curvature, there exists a unique Hermitian metric minimizing Calabi’s functional.

This leads us to make the following definition:

Definition. Let $L \rightarrow M$ be an ample holomorphic line bundle over a compact Kähler manifold. For $h \in P(M, L)$ and $m \leq q \leq 2m$, we say that $\mathcal{N}_{N,q,h}^{\text{crit}}$ (resp. $\mathcal{N}_{N,h}^{\text{crit}}$) is *asymptotically minimal* if for all $h_1 \neq h$ in $P(M, L)$, there exists $N_0 = N_0(h_1)$ such that

$$(12) \quad \mathcal{N}_{N,q,h}^{\text{crit}} < \mathcal{N}_{N,q,h_1}^{\text{crit}} \quad (\text{resp. } \mathcal{N}_{N,h}^{\text{crit}} < \mathcal{N}_{N,h_1}^{\text{crit}}) \quad \text{for } N \geq N_0.$$

Assuming (M, L) has a Hermitian metric h minimizing Calabi’s functional, we see from Corollary 1.4 that $\mathcal{N}_{N,q,h}^{\text{crit}}$ (resp. $\mathcal{N}_{N,h}^{\text{crit}}$) is asymptotically minimal as long as $\beta_{2q}(m) > 0$ (resp. $\beta_2(m) > 0$). Since we believe this to be the case for all dimensions, we state the following conjecture.

Conjecture 1.5. Let $h \in P(M, L)$, $m \leq q \leq 2m$. Then the following are equivalent:

- $\mathcal{N}_{N,h}^{\text{crit}}$ is asymptotically minimal,
- $\mathcal{N}_{N,q,h}^{\text{crit}}$ is asymptotically minimal (and hence $\mathcal{N}_{N,q',h}^{\text{crit}}$ is asymptotically minimal $\forall q'$),
- h minimizes Calabi’s functional.

In Lemma 6.1, we show that

$$(13) \quad \beta_{2q}(m) = \frac{1}{4\pi^{\binom{m+2}{2}}} \int_{\mathbf{S}_{m,q-m}} \gamma(H) |\det(2HH^* - |x|^2I)| \cdot e^{-\langle(H,x),(H,x)\rangle} dH dx,$$

where

$$\gamma(H) = \frac{1}{2}|H_{11}|^4 - 2|H_{11}|^2 + 1.$$

It is unfortunately difficult to determine from this formula whether $\beta_{2q}(m)$ is non-zero and (if so) what sign it has. In §6, we transform (13) to a rather complicated, but more elementary, integral (Lemma 6.2), which can be evaluated by a routine (but long) calculation. For low dimensions, we perform this calculation using Maple to obtain:

Theorem 1.6. *The constants $\beta_{2q}(m)$ are positive for $m \leq 5$, and hence Conjecture 1.5 is true for $\dim M \leq 5$.*

The calculation in dimension 5 was done by B. Baugher [2]. In particular, we have:

Corollary 1.7. *Suppose that $\dim M \leq 5$ and that L possesses a metric h for which the scalar curvature of $\omega_h = \frac{i}{2}\Theta_h$ is constant. Then h is the unique metric on L such that $\mathcal{N}_{N,h}^{\text{crit}}$ is asymptotically minimal.*

Thus, for instance, the Fubini-Study metric h on the hyperplane section bundle $\mathcal{O}(1) \rightarrow \mathbb{C}\mathbb{P}^m$ is the unique metric on $\mathcal{O}(1)$ such that $\mathcal{N}_{N,h}^{\text{crit}}$ is asymptotically minimal, at least for $m \leq 5$.

Additional evidence that $\beta_{2q}(m) > 0$ was found by B. Baugher [2]. By studying patterns in computer-assisted calculations of the terms arising from the computation of the integral in (13), Baugher discovered an identity for $\beta_{2q}(m)$ for dimensions $m \leq 5$ which he conjectures is valid in all dimensions. Baugher showed that this identity implies that $\beta_{2q}(m) > 0$ in all dimensions, and hence Baugher's conjecture implies Conjecture 1.5. Baugher's conjecture and identity are stated in Appendix 2.

We close the introduction with some comments on the organization of the paper. The proof of Theorems 1.1 and 1.2 is based on the Tian-Yau-Zelditch asymptotic expansion of the Szegő kernel $\Pi_N(z, w)$ [19, 23, 26, 28] and on formulas from our previous paper [11] for the density of critical points. We then need to evaluate the coefficients explicitly to obtain concrete results linking geometry to numbers of critical points. Once we know the leading coefficient is universal, we may calculate it for $\mathcal{O}(N) \rightarrow \mathbb{C}\mathbb{P}^m$ and this is done in §4 and Appendix 1. Unfortunately, the Fubini-Study metric is not useful for finding the sign of β_2 since it is impossible to separate out the topologically invariant terms from the Calabi functional for this metric. Hence in §6, we analyze instead the case of $M = \mathbb{C}\mathbb{P}^1 \times E^{m-1}$ for E an elliptic curve, where the relevant topological terms vanish. This leads to an explicit integral which we analyze by a variant of the Itzykson-Zuber formula in random matrix integrals.

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2. Background

Let $(L, h) \rightarrow M$ be a Hermitian holomorphic line bundle over a complex manifold M , and let $\nabla = \nabla_h$ be its Chern connection, i.e., the unique connection of type $(1, 0)$ on L compatible with both the metric and complex structure of L . Thus, it satisfies $\nabla''s = 0$ for any holomorphic section s where $\nabla = \nabla' + \nabla''$ is the splitting of the connection into its $L \otimes T^{*1,0}$, resp. $L \otimes T^{*0,1}$ parts. It follows that

$$(14) \quad \text{Crit}(s, h) = \{z : \nabla'_h s(z) = 0\}.$$

We denote by $\Theta_h = d\partial \log h = -\partial\bar{\partial} \log h$ the curvature of h and $\omega_h = \frac{i}{2}\Theta_h$.

We now introduce the Gaussian measures γ_h , called *Hermitian Gaussian measures* in [11], which we use exclusively in this paper. They are determined by the inner product

$$(15) \quad \langle s_1, s_2 \rangle = \int_M h(s_1(z), s_2(z)) dV_h(z)$$

on $H^0(M, L)$, where $dV_h = \frac{1}{m!}\omega_h^m$. By definition,

$$(16) \quad d\gamma_h(s) = \frac{1}{\pi^d} e^{-\|c\|^2} dc, \quad s = \sum_{j=1}^d c_j e_j,$$

where dc is Lebesgue measure and $\{e_j\}$ is an orthonormal basis for \mathcal{S} relative to \langle, \rangle .

Definition. The *expected distribution of critical points* of $s \in \mathcal{S} \subset H^0(M, L)$ with respect to γ_h is defined by

$$(17) \quad \mathbf{K}_h^{\text{crit}} = \int_{H^0(M, L)} \left[\sum_{z \in \text{Crit}(s, h)} \delta_z \right] d\gamma_h(s)$$

where δ_z is the Dirac point mass at z ; i.e.,

$$(18) \quad (\mathbf{K}_h^{\text{crit}}, \varphi) = \int_{H^0(M, L)} \left[\sum_{z: \nabla_h s(z)=0} \varphi(z) \right] d\gamma_h(s), \quad \varphi \in C^\infty(M).$$

The density of $\mathbf{K}_h^{\text{crit}}$ with respect to dV_h is denoted $\mathcal{K}_h^{\text{crit}}(z)$; i.e.,

$$\mathbf{K}_h^{\text{crit}} = \mathcal{K}_h^{\text{crit}}(z) dV_h.$$

2.1. Formulas for the expected distribution of critical points.

Let $(L, h) \rightarrow (M, \omega_h)$ be a Hermitian holomorphic line bundle on an m -dimensional compact Kähler manifold. We say that $H^0(M, L)$ has the 2-jet spanning property if all possible values and derivatives of order ≤ 2 are attained by the global sections $s \in H^0(M, L)$ at every point of M . In [11], we showed that if $H^0(M, L)$ has the 2-jet spanning property, then $\mathbf{K}_h^{\text{crit}}$ is absolutely continuous with respect to dV_h , and we obtained

an integral formula for $\mathcal{K}_h^{\text{crit}}(z_0)$ in terms of the Szegö kernel $\Pi(z, w)$ for $H^0(M, L)$ with respect to h . To describe this formula, we choose normal coordinates about $z_0 \in M$ and define the following matrices:

$$(19) \quad A(z_0) = (\nabla_{z_j} \nabla_{\bar{w}_{j'}} \Pi),$$

$$(20) \quad B(z_0) = \left[(\tau_{j'q'} \nabla_{z_j} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi) \quad (\nabla_{z_j} \Pi) \right],$$

$$(21) \quad C(z_0) = \begin{bmatrix} (\tau_{jq} \tau_{j'q'} \nabla_{z_q} \nabla_{z_j} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi) & (\tau_{jq} \nabla_{z_q} \nabla_{z_j} \Pi) \\ (\tau_{j'q'} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi) & \Pi \end{bmatrix},$$

$$(22) \quad \text{where } \tau_{jq} = \sqrt{2} \text{ if } j < q, \quad \tau_{jj} = 1, \\ 1 \leq j \leq m, \quad 1 \leq j \leq q \leq m, \quad 1 \leq j' \leq q' \leq m,$$

and where the Szegö kernel $\Pi = \Pi(z, w)$ and its derivatives are evaluated at $(z_0, 0; z_0, 0)$ (see §2.2 and (39)). In the above matrices, j, q index the rows, and j', q' index the columns. Note that A, B, C are $m \times m, m \times d_m, d_m \times d_m$ matrices, respectively, where

$$d_m = \dim_{\mathbb{C}}(\text{Sym}(m, \mathbb{C}) \times \mathbb{C}) = \frac{m^2 + m + 2}{2}.$$

We then let

$$(23) \quad \Lambda(z_0) = C(z_0) - B(z_0)^* A(z_0)^{-1} B(z_0).$$

The matrices A, B, C give the second moments of the joint probability distribution of the random variables $\nabla s(z_0)$ and $\nabla^2 s(z_0)$ on \mathcal{S} .

Theorem 2.1 ([11]). *Let $(L, h) \rightarrow M$ denote a positive holomorphic line bundle with the 2-jet spanning property on a compact complex manifold. Give M the Kähler form $\omega_h = \frac{i}{2} \Theta_h$ and volume form $dV_h = \frac{1}{m!} \omega_h^m$ induced from the curvature of L . Then the expected density relative to dV_h of critical points of random sections of $H^0(M, L)$ is given by*

$$\mathcal{K}_h^{\text{crit}}(z) = \frac{\pi^{-(\frac{m+2}{2})}}{\det A(z) \det \Lambda(z)} \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} |\det(HH^* - |x|^2 I)| \\ \cdot e^{-\langle \Lambda(z)^{-1}(H, x), (H, x) \rangle} dH dx.$$

Here, $H \in \text{Sym}(m, \mathbb{C})$ is a complex symmetric matrix, dH and dx denote Lebesgue measure, and Λ^{-1} is the Hermitian operator on the complex vector space $\text{Sym}(m, \mathbb{C}) \times \mathbb{C}$ described as follows:

Let $S^{jq}, 1 \leq j \leq q \leq m$, be the basis for $\text{Sym}(m, \mathbb{C})$ given by

$$(S^{jq})_{j'q'} = \begin{cases} \frac{1}{\sqrt{2}}(\delta_{jj'}\delta_{qq'} + \delta_{qj'}\delta_{jq'}) & \text{for } j < q \\ \delta_{jj'}\delta_{qq'} & \text{for } j = q. \end{cases}$$

I.e., for $j < q$, S^{jq} is the matrix with $\frac{1}{\sqrt{2}}$ in the jq and qj places and 0 elsewhere, while S^{jj} is the matrix with 1 in the jj place and 0 elsewhere.

We note that $\{S^{jq}\}$ is an orthonormal basis (over \mathbb{C}) for $\text{Sym}(m, \mathbb{C})$ with respect to the Hilbert-Schmidt Hermitian inner product

$$(24) \quad \langle S, T \rangle_{\text{HS}} = \text{Tr}(ST^*).$$

For $H = (H_{jq}) \in \text{Sym}(m, \mathbb{C})$, we have

$$(25) \quad H = \sum_{1 \leq j \leq q \leq m} \widehat{H}_{jq} E^{jq}, \quad \widehat{H}_{jq} = \tau_{jq} H_{jq},$$

where τ_{jq} is given by (22). Lebesgue measure dH (with respect to the Hilbert-Schmidt norm) is given by

$$dH = \prod_{j \leq q} d\text{Re} \widehat{H}_{jq} \wedge d\text{Im} \widehat{H}_{jq}.$$

Writing

$$\Lambda = \begin{bmatrix} (\Lambda_{jq}^{j'q'}) & (\Lambda_{jq}^0) \\ (\Lambda_0^{j'q'}) & \Lambda_0^0 \end{bmatrix},$$

we then define

$$(26) \quad \langle \Lambda(z)^{-1}(H, x), (H, x) \rangle = \sum (\Lambda^{-1})_{jq}^{j'q'} \widehat{H}_{jq} \overline{\widehat{H}_{j'q'}} + 2\text{Re} \sum (\Lambda^{-1})_{jq}^0 \widehat{H}_{jq} \bar{x} + (\Lambda^{-1})_0^0 |x|^2.$$

To study the asymptotics, we consider powers L^N and we let $\Pi_N(z, w)$ denote the Szegő kernel for $H^0(M, L^N)$; see (34)–(35). We shall use the following result.

Corollary 2.2. *With the same notation and assumptions as above, the density of the expected distribution $\mathbf{K}_{N,h}^{\text{crit}}$ of critical points of random sections $s_N \in H^0(M, L^N)$ relative to dV_h is given by*

$$\mathcal{K}_{N,h}^{\text{crit}}(z) = \frac{\pi^{-(\frac{m+2}{2})}}{\det A_N(z) \det \Lambda_N(z)} \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} |\det(HH^* - |x|^2 I)| \cdot e^{-\langle \Lambda_N(z)^{-1}(H, x), (H, x) \rangle} dH dx,$$

where

$$(27) \quad \Lambda_N(z_0) = C_N(z_0) - B_N(z_0)^* A_N(z_0)^{-1} B_N(z_0),$$

$$(28) \quad A_N(z_0) = \left[(\nabla_{z_j} \nabla_{\bar{w}_{j'}} \Pi_N) \right],$$

$$(29) \quad B_N(z_0) = \left[(\tau_{j'q'} \nabla_{z_j} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi_N) \quad (N \nabla_{z_j} \Pi_N) \right],$$

$$(30) \quad C_N(z_0) = \begin{bmatrix} (\tau_{jq} \tau_{j'q'} \nabla_{z_q} \nabla_{z_j} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi_N) & (\tau_{jq} N \nabla_{z_q} \nabla_{z_j} \Pi_N) \\ (\tau_{j'q'} N \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi_N) & N^2 \Pi_N \end{bmatrix},$$

$$\tau_{jq} = \sqrt{2} \quad \text{if } j < q, \quad \tau_{jj} = 1,$$

$$1 \leq j \leq m, \quad 1 \leq j \leq q \leq m, \quad 1 \leq j' \leq q' \leq m,$$

where Π_N and its derivatives are evaluated at $(z_0, 0; z_0, 0)$; see §2.2.1.

Proof. Rescale $z_j = \tilde{z}_j/\sqrt{N}$. Then the curvature of L^N is given by

$$\Theta_{h^N} = N\Theta_h = \frac{1}{2} \sum d\tilde{z}_j \wedge d\bar{\tilde{z}}_j,$$

so that the \tilde{z}_j are normal coordinates (at a point z_0) for the curvature of L^N . Apply Theorem 2.1, using the coordinates $\{\tilde{z}_j\}$ to obtain A, B, C, Λ . Since $d\tilde{V} = N^m dV$ and the transformation $(A, \Lambda) \mapsto (NA, N^2\Lambda)$ introduces a factor N^{-m} , we let $A_N = NA$, $B_N = N^{3/2}B$, $C_N = N^2C$ to obtain the desired formula. q.e.d.

We also have a formula for the density of critical points of specific Morse indices:

Theorem 2.3. *Under the above assumptions, the density relative to dV_h of the expected distribution $\mathbf{K}_{N,q,h}^{\text{crit}}$ of critical points of Morse index q of $\log \|s_N\|_h$ for random sections $s_N \in H^0(M, L^N)$ is given by*

$$\begin{aligned} \mathcal{K}_{N,q,h}^{\text{crit}}(z) = & \frac{\pi^{-\binom{m+2}{2}}}{\det A_N(z) \det \Lambda_N(z)} \int_{\mathbf{S}_{m,q-m}} |\det(HH^* - |x|^2 I)| \\ & \cdot e^{-\langle \Lambda_N(z)^{-1}(H,x), (H,x) \rangle} dH dx, \end{aligned}$$

where

$$\mathbf{S}_{m,k} = \{(H, x) \in \text{Sym}(m, \mathbb{C}) \times \mathbb{C} : \text{index}(HH^* - |x|^2 I) = k\}.$$

Proof. The case $N = 1$ is given as Theorem 6 in [11]. The general case follows immediately by rescaling as in the proof of Corollary 2.2. q.e.d.

Recall that the index of a nonsingular Hermitian matrix is the number of its negative eigenvalues, and the Morse index of a nondegenerate critical point of a real-valued function is the index of its (real) Hessian.

2.2. The Szegö kernel. As in our previous work, it is useful to lift the analysis on positive line bundles $L \rightarrow M$ to the associated principal S^1 bundle $X \rightarrow M$. Sections then become scalar functions and it is simpler to formulate various asymptotic properties for powers L^N [3, 4]. The same analysis is also useful for general line bundles although the asymptotic results no longer hold.

Given a holomorphic line bundle L and a Hermitian metric h on L , we obtain a Hermitian metric h^* on the dual line bundle L^* and we define the associated circle bundle by $X = \{\lambda \in L^* : \|\lambda\|_{h^*} = 1\}$. Thus, X is the boundary of the disc bundle $D = \{\lambda \in L^* : \rho(\lambda) > 0\}$, where $\rho(\lambda) = 1 - \|\lambda\|_{h^*}^2$. When (L, h) is a positive line bundle, the disc bundle D is strictly pseudoconvex in L^* ; hence X inherits the structure of a strictly pseudoconvex CR manifold. When L is negative, as is the case for the line bundles relevant to string theory, X is pseudo-concave. We

endow X with the contact form $\alpha = -i\partial\rho|_X = i\bar{\partial}\rho|_X$ and the associated volume form

$$(31) \quad dV_X = \frac{1}{m!} \alpha \wedge (d\alpha)^m = \alpha \wedge \pi^* dV_M.$$

We define the Hardy space $\mathcal{H}^2(X) \subset \mathcal{L}^2(X)$ of square-integrable CR functions on X , i.e., functions that are annihilated by the Cauchy-Riemann operator $\bar{\partial}_b$ and are \mathcal{L}^2 with respect to the inner product

$$(32) \quad \langle F_1, F_2 \rangle = \frac{1}{2\pi} \int_X F_1 \overline{F_2} dV_X, \quad F_1, F_2 \in \mathcal{L}^2(X).$$

We let $r_\theta x = e^{i\theta}x$ ($x \in X$) denote the S^1 action on X and denote its infinitesimal generator by $\frac{\partial}{\partial\theta}$. The S^1 action on X commutes with $\bar{\partial}_b$; hence $\mathcal{H}^2(X) = \bigoplus_{N=0}^\infty \mathcal{H}_N^2(X)$ where $\mathcal{H}_N^2(X) = \{F \in \mathcal{H}^2(X) : F(r_\theta x) = e^{iN\theta}F(x)\}$. A section s_N of L^N determines an equivariant function \hat{s}_N on L^* by the rule

$$\hat{s}_N(\lambda) = (\lambda^{\otimes N}, s_N(z)), \quad \lambda \in L_z^*, \quad z \in M,$$

where $\lambda^{\otimes N} = \lambda \otimes \dots \otimes \lambda$. We henceforth restrict \hat{s} to X and then the equivariance property takes the form $\hat{s}_N(r_\theta x) = e^{iN\theta}\hat{s}_N(x)$. The map $s \mapsto \hat{s}$ is a unitary equivalence between $H^0(M, L^N)$ and $\mathcal{H}_N^2(X)$.

We let e_L be a nonvanishing local section, or local frame, of L . As above, we write

$$(33) \quad \|e_L(z)\|_h^2 = e^{-K(z, \bar{z})}.$$

Thus, a positive line bundle L induces the Kähler form $\omega_h = \frac{i}{2}\partial\bar{\partial}K$ with Kähler potential K .

The Szegő kernel $\Pi_N(x, y)$ is the kernel of the orthogonal projection $\Pi_N : \mathcal{L}^2(X) \rightarrow \mathcal{H}_N^2(X)$; it is defined by

$$(34) \quad \Pi_N F(x) = \int_X \Pi_N(x, y) F(y) dV_X(y), \quad F \in \mathcal{L}^2(X).$$

Let $\{s_j^N = f_j e_L^{\otimes N} : j = 1, \dots, d_N\}$ be an orthonormal basis for $H^0(M, L^N)$. Then $\{\hat{s}_j^N\}$ is an orthonormal basis of $\mathcal{H}^2(X)$, and the Szegő kernel can be written in the form

$$(35) \quad \Pi_N(x, y) = \sum_{j=1}^{d_N} \hat{s}_j^N(x) \overline{\hat{s}_j^N(y)}.$$

It is the lift of the section

$$(36) \quad \tilde{\Pi}_N(z, \bar{w}) := F_N(z, \bar{w}) e_L^{\otimes N}(z) \otimes \overline{e_L^{\otimes N}(w)},$$

where

$$(37) \quad F_N(z, \bar{w}) = \sum_{j=1}^{d_N} f_j(z) \overline{f_j(w)}.$$

We let (z, θ) denote the coordinates of the point $x = e^{i\theta} \|e_L(z)\|_h e_L^*(z) \in X$. The equivariant lift of a section $s = f e_L^{\otimes N} \in H^0(M, L^N)$ is given explicitly by

$$(38) \quad \hat{s}(z, \theta) = e^{iN\theta} \|e_L^{\otimes N}\|_h f(z) = e^{N[-\frac{1}{2}K(z, \bar{z}) + i\theta]} f(z).$$

The Szegö kernel is then given by

$$(39) \quad \Pi_N(z, \theta; w, \varphi) = e^{N[-\frac{1}{2}K(z, \bar{z}) - \frac{1}{2}K(w, \bar{w}) + i(\theta - \varphi)]} F_N(z, \bar{w}).$$

2.2.1. The connection. We denote by $H = \ker \alpha$ and obtain a splitting $T_X = H \oplus \mathbb{C} \frac{\partial}{\partial \theta}$ into horizontal and vertical spaces. The Chern connection ∇ on L^N then lifts to X as the horizontal derivative d^H , i.e.

$$(40) \quad (\nabla s_N)^\wedge = d^H \hat{s}_N.$$

To describe the connection explicitly, we choose local holomorphic coordinates $\{z_1, \dots, z_m\}$ in M , and we write

$$\nabla = \nabla' + \nabla'', \quad \nabla' s_N = \sum dz_j \otimes \nabla_{z_j} s_N, \quad \nabla'' s_N = \sum d\bar{z}_j \otimes \nabla_{\bar{z}_j} s_N.$$

In particular, $(\nabla'' s_N)^\wedge = \bar{\partial}_b \hat{s}_N$, which vanishes when the section s_N is holomorphic, or equivalently, when $\hat{s}_N \in \mathcal{H}_N^2(X)$.

For a C^∞ section $s_N = f e_L^{\otimes N}$ of L^N , we have

$$(41) \quad \begin{aligned} \nabla_{z_j} s_N &= \left(\frac{\partial f}{\partial z_j} - N f \frac{\partial K}{\partial z_j} \right) e_L^{\otimes N} = e^{NK} \frac{\partial}{\partial z_j} (e^{-NK} f) e_L^{\otimes N}, \\ \nabla_{\bar{z}_j} s_N &= \frac{\partial f}{\partial \bar{z}_j} e_L^{\otimes N} \quad (\nabla \text{ is of type } (1, 0)). \end{aligned}$$

We also write

$$(42) \quad d^H \Pi_N(z, \theta; w, \varphi) = \sum dz_j \otimes \nabla_{z_j} \Pi_N + \sum d\bar{w}_j \otimes \nabla_{\bar{w}_j} \Pi_N,$$

where d^H is the horizontal derivative on $X \times X$. (We used the fact that the horizontal derivatives of Π_N with respect to the \bar{z}_j and w_j variables vanish.) By (39)–(41), we have

$$(43) \quad \nabla_{z_j} \Pi_N = e^{N[-\frac{1}{2}K(z, \bar{z}) - \frac{1}{2}K(w, \bar{w}) + i(\theta - \varphi)]} \left(\frac{\partial}{\partial z_j} - N \frac{\partial K}{\partial z_j}(z, \bar{z}) \right) F_N(z, \bar{w}),$$

$$(44) \quad \nabla_{\bar{w}_j} \Pi_N = e^{N[-\frac{1}{2}K(z, \bar{z}) - \frac{1}{2}K(w, \bar{w}) + i(\theta - \varphi)]} \left(\frac{\partial}{\partial \bar{w}_j} - N \frac{\partial K}{\partial \bar{w}_j}(w, \bar{w}) \right) F_N(z, \bar{w}).$$

3. Alternate formulas for the density of critical points

The integrals in Theorems 2.1–2.3 are difficult to evaluate because of the absolute value sign, which prevents application of Wick methods. To compute the densities, we shall replace our integral by another one which can be evaluated by residue calculus in certain cases. This new integral is given by the following lemma:

Lemma 3.1. *Let Λ be a positive definite Hermitian operator on $\text{Sym}(m, \mathbb{C}) \times \mathbb{C}$. Then*

$$\begin{aligned} & \frac{1}{\pi^{d_m} \det \Lambda} \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} |\det(HH^* - |x|^2 I)| e^{-\langle \Lambda^{-1}(H, x), (H, x) \rangle} dH dx \\ &= \frac{(-i)^{m(m-1)/2}}{(2\pi)^m \prod_{j=1}^m j!} \lim_{\varepsilon' \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^m} d\lambda \int_{\mathbb{R}^m} d\xi \int_{U(m)} dg \\ & \quad \cdot \frac{\Delta(\xi) \Delta(\lambda) |\prod_j \lambda_j| e^{i\langle \xi, \lambda \rangle} e^{-\varepsilon|\xi|^2 - \varepsilon'|\lambda|^2}}{\det \left[i\widehat{D}(\xi)\rho(g)\Lambda\rho(g)^* + I \right]}, \end{aligned}$$

where

- $\Delta(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j)$,
- dg is unit mass Haar measure on $U(m)$,
- $\widehat{D}(\xi)$ is the Hermitian operator on $\text{Sym}(m, \mathbb{C}) \oplus \mathbb{C}$ given by

$$\widehat{D}(\xi)((H_{jk}), x) = \left(\left(\frac{\xi_j + \xi_k}{2} H_{jk} \right), - \left(\sum_{q=1}^m \xi_q \right) x \right),$$

- ρ is the representation of $U(m)$ on $\text{Sym}(m, \mathbb{C}) \oplus \mathbb{C}$ given by

$$\rho(g)(H, x) = (gHg^t, x).$$

The integrand is analytic in ξ, g but rather complicated. Its principal features are:

- $\Delta(\xi), \Delta(\lambda)$ are homogeneous polynomials of degree $m(m-1)/2$, and $|\prod_j \lambda_j|$ is homogeneous of degree m .
- $P_{g,z}(\xi) = \det \left[i\widehat{D}(\xi)\rho(g)\Lambda(z)\rho(g)^* + I \right]$ is a (family of) polynomial(s) in ξ of degree $m(m+1)/2 + 1$ with no real zeros $\xi \in \mathbb{R}^m$.

The proof of Lemma 3.1 is given in §3.1 below.

As a consequence, we have the following alternative formula for the expected critical point density:

Theorem 3.2. *Under the hypotheses of Theorem 2.1 and notation of Lemma 3.1, the density of the expected distribution of critical points of sections of $H^0(M, L^N)$ is also given by:*

$$\begin{aligned} \mathcal{K}_{N,h}^{\text{crit}}(z) &= \frac{c_m}{\det A_N} \lim_{\varepsilon' \rightarrow 0^+} \int_{\mathbb{R}^m} d\lambda \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^m} d\xi \int_{U(m)} dg \\ & \quad \cdot \frac{\Delta(\xi) \Delta(\lambda) |\prod_j \lambda_j| e^{i\langle \xi, \lambda \rangle} e^{-\varepsilon|\xi|^2 - \varepsilon'|\lambda|^2}}{\det \left[i\widehat{D}(\xi)\rho(g)\Lambda_N(z)\rho(g)^* + I \right]}, \end{aligned}$$

where

$$c_m = \frac{(-i)^{m(m-1)/2}}{2^m \pi^{2m} \prod_{j=1}^m j!}.$$

Proof. The formula follows by combining Corollary 2.2 and Lemma 3.1. q.e.d.

In §4 we shall use Theorem 3.2 to calculate the density of critical points for random sections $s_N \in H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(N))$ of the N -th power of the hyperplane bundle. In this case the $U(m)$ integral drops out, and the integral can be evaluated as an iterated integral without the Gaussian factor $e^{-\varepsilon|\xi|^2 - \varepsilon'|\lambda|^2}$.

We also have an alternative formula for the Morse index densities, which follows by a similar argument (given in §3.2):

Theorem 3.3. *Under the above assumptions, the density of the expected distribution of critical points of Morse index q of $\log \|s_N\|_h$ is also given by:*

$$\mathcal{K}_{N,q,h}^{\text{crit}}(z) = \frac{m! c_m}{\det A_N} \lim_{\varepsilon' \rightarrow 0^+} \int_{Y_{2m-q}} d\lambda \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^m} d\xi \int_{U(m)} dg \cdot \frac{\Delta(\xi) \Delta(\lambda) \prod_j |\lambda_j| e^{i\langle \xi, \lambda \rangle} e^{-\varepsilon|\xi|^2 - \varepsilon'|\lambda|^2}}{\det [i\widehat{D}(\xi)\rho(g)\Lambda_N(z)\rho(g)^* + I]},$$

where

$$Y_p = \{\lambda \in \mathbb{R}^m : \lambda_1 > \dots > \lambda_p > 0 > \lambda_{p+1} > \dots > \lambda_m\}.$$

3.1. Proof of Lemma 3.1. We write

$$(45) \quad \mathcal{I} = \frac{1}{\pi^{dm} \det \Lambda} \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} |\det(HH^* - |x|^2 I)| \cdot \exp(-\langle \Lambda^{-1}(H, x), (H, x) \rangle) dH dx.$$

Here, H (previously denoted by H') is a complex $m \times m$ symmetric matrix, so $H^* = \overline{H}$. The proof is basically to rewrite (45) using the Itzykson-Zuber integral and Gaussian integration.

We first observe that

$$\mathcal{I} = \lim_{\varepsilon' \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon, \varepsilon'},$$

where $\mathcal{I}_{\varepsilon, \varepsilon'}$ is the absolutely convergent integral,

$$(46) \quad \mathcal{I}_{\varepsilon, \varepsilon'} = \frac{1}{(2\pi)^{m^2} \pi^{dm} \det \Lambda} \int_{\mathcal{H}_m} \int_{\mathcal{H}_m} \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} |\det P| \cdot e^{-\varepsilon \text{Tr} \Xi^* \Xi - \varepsilon' \text{Tr} P^* P} e^{i\langle \Xi, P - HH^* + |x|^2 I \rangle} \cdot \exp(-\langle \Lambda^{-1}(H, x), (H, x) \rangle) dH dx dP d\Xi.$$

Here, \mathcal{H}_m is the space of $m \times m$ Hermitian matrices. Absolute convergence is guaranteed by the Gaussian factors in each variable (H, x, Ξ, P) . If the $d\Xi$ integral is done first, we obtain a dual Gaussian which converges (in the sense of tempered distributions) to the delta function

$\delta_{HH^* - \frac{1}{2}|x|^2}(P)$ as $\varepsilon \rightarrow 0$. Then, as $\varepsilon' \rightarrow 0$, the dP integral then evaluates the integrand at $P = HH^* - |x|^2I$ and we retrieve the original integral \mathcal{I} .

We next conjugate P in (46) to a diagonal matrix $D(\lambda)$ with $\lambda = (\lambda_1, \dots, \lambda_m)$ by an element $h \in U(m)$. Recalling that

$$(47) \quad \int_{\mathcal{H}_m} \varphi(P) dP = c'_m \int_{\mathbb{R}^m} \int_{U(m)} \varphi(hD(\lambda)h^*) \Delta(\lambda)^2 dh d\lambda, \quad c'_m = \frac{(2\pi)^{\binom{m}{2}}}{\prod_{j=1}^m j!}$$

(see for example [29, (1.9)]), we then obtain

$$\begin{aligned} \mathcal{I}_{\varepsilon, \varepsilon'} = & \frac{c'_m}{(2\pi)^{m^2} \pi^{d_m} \det \Lambda} \int_{U(m)} \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} \int_{\mathcal{H}_m} \int_{\mathbb{R}^m} |\det(D(\lambda))| \\ & \cdot e^{-\varepsilon(\text{Tr} D(\lambda)^* D(\lambda) + \text{Tr} \Xi^* \Xi)} e^{i\langle \Xi, hD(\lambda)h^* + |x|^2I - H^*H \rangle} \Delta(\lambda)^2 \\ & \cdot \exp(-\langle \Lambda^{-1}(H, x), (H, x) \rangle) d\lambda d\Xi dH dx dh. \end{aligned}$$

Since the factor $\int_{U(m)} e^{i\langle \Xi, hD(\lambda)h^* \rangle} dh$ is invariant under the conjugation $\Xi \rightarrow g^* \Xi g$ with $g \in U(m)$, we apply the same identity (47) in the Ξ variable. We write $\Xi = g^{-1} D(\xi) g$ where $D(\xi)$ is diagonal. This replaces $d\Xi$ by $\Delta(\xi)^2 d\xi$. The inner product is bi-invariant so we may transfer the conjugation to HH^* . We thus obtain:

$$(48) \quad \begin{aligned} \mathcal{I}_{\varepsilon, \varepsilon'} = & \frac{(c'_m)^2}{(2\pi)^{m^2} \pi^{d_m} \det \Lambda} \int_{U(m)} dh \int_{U(m)} dg \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} dH dx \\ & \cdot \int_{\mathbb{R}^m} d\lambda \int_{\mathbb{R}^m} d\xi |\det(D(\lambda))| \Delta(\lambda)^2 \Delta(\xi)^2 e^{-\varepsilon(|\xi|^2 + |\lambda|^2)} \\ & \cdot e^{i\langle D(\xi), hD(\lambda)h^* + |x|^2I - gHH^*g^* \rangle} e^{-\langle \Lambda^{-1}(H, x), (H, x) \rangle}. \end{aligned}$$

Next we recognize the integral $\int_{U(m)} e^{i\langle D(\xi), hD(\lambda)h^* \rangle} dh$ as the well-known Itzykson-Zuber-Harish-Chandra integral [15] (cf. [29]):

$$(49) \quad J(D(\lambda), D(\xi)) = (-i)^{m(m-1)/2} \left(\prod_{j=1}^{m-1} j! \right) \frac{\det[e^{i\lambda_j \xi_k}]_{j,k}}{\Delta(\lambda) \Delta(\xi)}.$$

We note that both numerator and denominator are anti-symmetric in ξ_j and λ_j under permutation, so that the ratio is well-defined.

We substitute (49) into (48) and expand

$$\det[e^{i\xi_j \lambda_k}]_{j,k} = \sum_{\sigma \in S_m} (-1)^\sigma e^{i\langle \xi, \sigma(\lambda) \rangle},$$

obtaining a sum of $m!$ integrals. However, by making the change of variables $\lambda' = \sigma(\lambda)$ and noting that $\Delta(\sigma(\lambda)) = (-1)^\sigma \Delta(\lambda)$, we see that

these integrals are equal, and (48) then becomes

$$(50) \quad \mathcal{I}_{\varepsilon, \varepsilon'} = \frac{(-i)^{m(m-1)/2}}{(2\pi)^m (\prod_{j=1}^m j!) \pi^{d_m} \det \Lambda} \int_{U(m)} dg \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} dH dx \\ \cdot \int_{\mathbb{R}^m} d\lambda \int_{\mathbb{R}^m} d\xi \Delta(\lambda) \Delta(\xi) |\det(D(\lambda))| e^{-\varepsilon(|\xi|^2 + |\lambda|^2)} \\ \cdot e^{i\langle \lambda, \xi \rangle} e^{i\langle D(\xi), |x|^2 I - gHH^*g^* \rangle - \langle \Lambda^{-1}(H, x), (H, x) \rangle}.$$

Further we observe that the $dHdx$ integral is a Gaussian integral. We simplify the phase by noting that

$$\langle D(\xi), gHH^*g^* - |x|^2 I \rangle = \text{Tr}(D(\xi)gHg^t \bar{g}H^*g^*) - \text{Tr}D(\xi) |x|^2 \\ = \left\langle \widehat{D}(\xi)\rho(g)(H, x), \rho(g)(H, x) \right\rangle,$$

where $\widehat{D}(\xi)$ and $\rho(g)$ are as in the statement of the theorem. Thus,

$$(51) \quad \frac{1}{\pi^{d_m} \det \Lambda} \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} e^{i\langle D(\xi), |x|^2 I - gHH^*g^* \rangle - \langle \Lambda^{-1}(H, x), (H, x) \rangle} dH dx \\ = \frac{1}{\det \Lambda \det \left[i\rho(g)^* \widehat{D}(\xi)\rho(g) + \Lambda^{-1} \right]} \\ = \frac{1}{\det \left[i\rho(g)^* \widehat{D}(\xi)\rho(g)\Lambda + I \right]} \\ = \frac{1}{\det \left[i\widehat{D}(\xi)\rho(g)\Lambda\rho(g)^* + I \right]}.$$

Substituting (51) into (50), we obtain the desired formula. q.e.d.

3.2. Proof of Theorem 3.3. The proof is similar to that of Theorem 3.2 above, with the modifications provided by Theorem 2.3. The proof of Lemma 3.1 shows that

$$\frac{1}{\det \Lambda} \int_{\mathbf{S}_{m,k}} |\det(HH^* - |x|^2 I)| e^{-\langle \Lambda^{-1}(H, x), (H, x) \rangle} dH dx \\ = \frac{(-i)^{m(m-1)/2}}{(2\pi)^m \prod_{j=1}^m j!} \lim_{\varepsilon' \rightarrow 0^+} \int_{Y'_{m-k}} d\lambda \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^m} d\xi \int_{U(m)} dg \\ \cdot \frac{\Delta(\xi) \Delta(\lambda) |\prod_j \lambda_j| e^{i\langle \xi, \lambda \rangle} e^{-\varepsilon|\xi|^2 - \varepsilon'|\lambda|^2}}{\det \left[i\widehat{D}(\xi)\rho(g)\Lambda\rho(g)^* + I \right]},$$

where Y'_p denotes the set of points in \mathbb{R}^m with exactly p coordinates positive. Since the integrand on the right is invariant under identical simultaneous permutations of the ξ_j and the λ_j , it follows that the integral equals $m!$ times the corresponding integral over Y_{m-k} . The desired formula then follows from Theorem 2.3. q.e.d.

4. Exact formula for $\mathbb{C}\mathbb{P}^m$

To illustrate our results for fixed N , we compute the density $\mathcal{K}_{N,q}^{\text{crit}}(z)$ of the expected distribution of critical points of Morse index q of $\log \|s_N\|_{h^N}$ for random sections $s_N \in H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(N))$, where h^N is the Fubini-Study metric on $\mathcal{O}(N)$. Here, the probability measure on $H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(N))$ is the Gaussian measure induced from h^N and the volume form $V = \frac{1}{m!} \omega_{\text{FS}}^m$ on $\mathbb{C}\mathbb{P}^m$. Since this Hermitian metric and Gaussian measure are invariant under the $\text{SU}(m+1)$ action on $\mathbb{C}\mathbb{P}^m$, the density is independent of the point $z \in \mathbb{C}\mathbb{P}^m$, and hence the expected number of critical points of Morse index q is given by

$$(52) \quad \mathcal{N}_{N,q}^{\text{crit}}(\mathbb{C}\mathbb{P}^m) = \frac{\pi^m}{m!} \mathcal{K}_{N,q}^{\text{crit}}(z).$$

These numbers turn out to be rational functions of N given by the following integral formula:

Proposition 4.1. *The expected number of critical points of Morse index q for random sections $s_N \in H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(N))$ is given by*

$$\mathcal{N}_{N,q}^{\text{crit}}(\mathbb{C}\mathbb{P}^m) = \frac{2^{\frac{m^2+m+2}{2}}}{\prod_{j=1}^m j!} \frac{(N-1)^{m+1}}{(m+2)N-2} \int_{Y_{2m-q}} d\lambda \left| \prod_{j=1}^m \lambda_j \right| \Delta(\lambda) \cdot e^{-\sum_{j=1}^m \lambda_j} \cdot \begin{cases} e^{(m+2-2/N)\lambda_m} & \text{for } q > m \\ 1 & \text{for } q = m \end{cases},$$

for $N \geq 2$, where Y_{2m-q} is as in Theorem 3.3.

Thus, the expected number $\mathcal{N}_{N,m}^{\text{crit}}(\mathbb{C}\mathbb{P}^m)$ of critical points of minimum Morse index is of the form $\kappa_m \frac{(N-1)^{m+1}}{(m+2)N-2}$, for some constant $\kappa_m > 0$. It is also easy to see from the form of the above integral that the expected number $\mathcal{N}_{N,q}^{\text{crit}}(\mathbb{C}\mathbb{P}^m)$ of critical points of each Morse index q is a rational function of N . In Appendix 1, we give explicit formulas for $\mathcal{N}_{N,q}^{\text{crit}}(\mathbb{C}\mathbb{P}^m)$ in low dimensions obtained by using Maple to evaluate the integral in Proposition 4.1.

4.1. Itzykson-Zuber formula on $\mathbb{C}\mathbb{P}^m$. The following formula derived from Theorem 3.3 is the starting point for our proof of Proposition 4.1.

Lemma 4.2. *The expected critical point density for random sections $s_N \in H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(N))$ of Morse index q is given by*

$$\begin{aligned} & \mathcal{K}_{N,q}^{\text{crit}}(z) \\ &= i^{m+1} \frac{m! |c_m|}{N^m} \lim_{\varepsilon' \rightarrow 0^+} \int_{Y_{2m-q}} d\lambda \left| \prod_j \lambda_j \right| \Delta(\lambda) e^{-\varepsilon' |\lambda|^2} \\ & \cdot \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^m} \frac{\Delta(\xi) e^{i\langle \lambda, \xi \rangle} e^{-\varepsilon |\xi|^2} d\xi}{(N^2 \sum \xi_j + i) \prod_{1 \leq j \leq k \leq m} \{i - N(N-1)(\xi_j + \xi_k)\}}, \end{aligned}$$

where c_m and Y_{2m-q} are as in Theorems 3.2 and 3.3.

Proof. Since the critical point density $\mathcal{K}_{N,q}^{\text{crit}}$ is constant, it suffices to compute it at $z = 0 \in \mathbb{C}^m \subset \mathbb{C}\mathbb{P}^m$, using the local frame e_L corresponding to the homogeneous (linear) polynomial z_0 . We recall that the Szegő kernel is given by

$$\Pi_{H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(N))}(z, w) = \frac{(N+m)!}{\pi^m N!} (1 + z \cdot \bar{w})^N e_L(z) \otimes \overline{e_L(w)}.$$

(See, for example, [3, §1.3].) Since the formula in Theorem 2.1 is invariant when the Szegő kernel is multiplied by a constant, we can replace the above by the *normalized Szegő kernel*

$$(53) \quad \tilde{\Pi}_N(z, w) := (1 + z \cdot \bar{w})^N$$

in our computation.

We notice that

$$\begin{aligned} K(z) &= -\log \|e_L(z)\|_h^2 = \log(1 + \|z\|^2), \\ K(0) &= \frac{\partial K}{\partial z}(0) = \frac{\partial^2 K}{\partial^2 z}(0) = 0. \end{aligned}$$

Hence when computing the (normalized) matrices $\tilde{A}_N, \tilde{B}_N, \tilde{C}_N$, we can take the usual derivatives of $\tilde{\Pi}_N$. Indeed, we have

$$\begin{aligned} \frac{\partial \tilde{\Pi}_N}{\partial z_j} &= N(1 + z \cdot \bar{w})^{N-1} \bar{w}_j, \\ \frac{\partial^2 \tilde{\Pi}_N}{\partial z_j \partial \bar{w}_{j'}} &= \delta_{jj'} N(1 + z \cdot \bar{w})^{N-1} + N(N-1)(1 + z \cdot \bar{w})^{N-2} z_{j'} \bar{w}_j, \\ \frac{\partial^4 \tilde{\Pi}_N}{\partial z_j \partial z_q \partial \bar{w}_{j'} \partial \bar{w}_{q'}}(0, 0) &= N(N-1)(\delta_{jj'} \delta_{qq'} + \delta_{j'q} \delta_{jq'}). \end{aligned}$$

It follows that

$$(54) \quad \tilde{A}_N = NI, \quad \tilde{B}_N = 0, \quad \tilde{\Lambda}_N = \tilde{C}_N = \begin{pmatrix} 2N(N-1)\hat{I} & 0 \\ 0 & N^2 \end{pmatrix},$$

where \hat{I} is the identity matrix of rank $\binom{m+1}{2}$.

The stated formula now follows from Theorem 3.3 by observing that $\rho(g)\tilde{\Lambda}_N \rho(g)^* = \tilde{\Lambda}_N$, and

$$\det \left[i\widehat{D}(\xi)\tilde{\Lambda}_N + I \right] = (-i)^{\frac{m^2+m+2}{2}} \left(N^2 \sum \xi_j + i \right) \cdot \prod_{1 \leq j \leq k \leq m} \{i - N(N-1)(\xi_j + \xi_k)\}.$$

q.e.d.

4.2. Evaluating the inner integral by residues. To complete the proof of Proposition 4.1, we must evaluate the $d\xi$ integral in Lemma 4.2. We begin by writing

$$(55) \quad \mathcal{K}_{N,q}^{\text{crit}}(z) = i^{m+1} m! |c_m| \lim_{\varepsilon' \rightarrow 0^+} \int_{Y_{2m-q}} d\lambda \left| \prod_j \lambda_j \right| \Delta(\lambda) e^{-\varepsilon' |\lambda|^2} \mathcal{I}_{N,\lambda},$$

where

$$\mathcal{I}_{N,\lambda} := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{N^m} \int_{\mathbb{R}^m} \frac{\Delta(\xi) e^{i\langle \lambda, \xi \rangle} e^{-\varepsilon |\xi|^2} d\xi}{(N^2 \sum \xi_j + i) \prod_{1 \leq j \leq k \leq m} \{i - N(N-1)(\xi_j + \xi_k)\}}.$$

To simplify the constant factors, we make the redefinitions $\xi_j = (t_j + i)/2N(N-1)$ and $\lambda_j \rightarrow 2N(N-1)\lambda_j$, after which (55) holds with

$$\mathcal{I}_{N,\lambda} = (-1)^{\frac{m(m+1)}{2}} 2^{\frac{(m+1)(m+2)}{2}} \frac{(N-1)^{m+1}}{N} e^{-\sum \lambda_j} \mathcal{I}(\lambda; c),$$

where

$$(56) \quad \mathcal{I}(\lambda; c) = \lim_{\varepsilon \rightarrow 0^+} \int_{(\mathbb{R}-i)^m} \frac{\Delta(t) e^{i\langle \lambda, t \rangle} e^{-\varepsilon \sum |t_j|^2}}{(\sum t_j + ic) \prod_{1 \leq j \leq k \leq m} (t_j + t_k)} dt,$$

$$(57) \quad c = m + 2 - 2/N.$$

Recalling the definition of c_m in the statement of Theorem 3.2, we therefore have:

$$(58) \quad \mathcal{K}_{N,q}^{\text{crit}}(z) = i^{(m+1)^2} \frac{2^{\frac{m^2+m+2}{2}} (N-1)^{m+1}}{\pi^{2m} \prod_{j=1}^{m-1} j!} \frac{(N-1)^{m+1}}{N} \cdot \lim_{\varepsilon' \rightarrow 0^+} \int_{Y_{2m-q}} d\lambda \left| \prod_j \lambda_j \right| \Delta(\lambda) e^{-\varepsilon' |\lambda|^2} e^{-\sum \lambda_j} \mathcal{I}(\lambda; c),$$

where $\mathcal{I}(\lambda; c)$ is given by (56)–(57). Proposition 4.1 now follows from (52), (58), and the following lemma with $p = 2m - q$, $c = m + 2 - 2/N$.

Lemma 4.3. *Let $0 \leq p \leq m$ and let $c > 0$. Then for*

$$\lambda_1 > \cdots > \lambda_p > 0 > \lambda_{p+1} > \cdots > \lambda_m,$$

we have

$$\mathcal{I}(\lambda; c) = \begin{cases} i^{m^2-1} \frac{\pi^m}{c} e^{c\lambda_m} & \text{for } p < m \\ i^{m^2-1} \frac{\pi^m}{c} & \text{for } p = m \end{cases},$$

where $\mathcal{I}(\lambda, c)$ is given by (56).

We note that $\mathcal{I}(\lambda, c)$, resp. $i\mathcal{I}(\lambda, c)$, is positive if m is odd, resp. even, and thus by monotone convergence we can set $\varepsilon' = 0$ in (58).

Proof. We let

$$(59) \quad \mathcal{I}(\lambda, t; c) = \frac{\Delta(t) e^{i\langle \lambda, t \rangle}}{(\sum t_j + ic) \prod_{1 \leq j \leq k \leq m} (t_j + t_k)} dt, \quad \text{for } c > 0.$$

We note that $\int_{(\mathbb{R}-i)^m} \mathcal{I}(\lambda, t; c) dt$ is a tempered distribution (in λ). Furthermore, the map

$$(\varepsilon_1, \dots, \varepsilon_m) \mapsto \int_{(\mathbb{R}-i)^m} \mathcal{I}(\lambda, t; c) e^{-\sum \varepsilon_j |t_j|^2} dt$$

is a continuous map from $[0, +\infty)^m$ to the tempered distributions. Here, $(\mathbb{R} - i)^m$ is the change in the contour \mathbb{R}^m obtained by translating $\mathbb{R} \rightarrow \mathbb{R} - i$ in each factor. Hence

$$(60) \quad \mathcal{I}(\lambda; c) = \int_{(\mathbb{R}-i)^m} \mathcal{I}(\lambda, t; c) dt = \lim_{\varepsilon_m \rightarrow 0^+} \cdots \lim_{\varepsilon_1 \rightarrow 0^+} \int_{(\mathbb{R}-i)^m} \mathcal{I}(\lambda, t; c) e^{-\sum \varepsilon_j |t_j|^2} dt.$$

We now use (60) to evaluate $\mathcal{I}(\lambda; c)$ by iterated residues. We first suppose that $p > 0$, and we start by doing the integral over t_1 . Since the t_1 integral is absolutely convergent when $\varepsilon_1 = 0$, we can set $\varepsilon_1 = 0$ and do the integral by residues. If $p > 0$ we close the contour in the upper half plane, and pick up poles at $t_1 = 0$, and at $t_1 = -t_j$ for $j \neq 1$. The pole at $t_1 = -ic - \sum_{j \neq 1} t_j$ is below the contour.

The residue of $\mathcal{I}(\lambda, t; c)$ at the pole $t_1 = 0$ is

$$(61) \quad \frac{(-1)^{m-1}}{2} \mathcal{I}(\lambda_2, \dots, \lambda_m, t_2, \dots, t_m; c).$$

The residue at the pole $t_1 = -t_2$ is

$$\begin{aligned} & \frac{\pm e^{i[(\lambda_2 - \lambda_1)t_2 + \lambda_3 t_3 + \dots + \lambda_m t_m]} 2t_2 (t_2 + t_3) \cdots (t_2 + t_m) \Delta(t_2, \dots, t_m)}{(t_3 + \dots + t_m + ci) 2t_2 (-t_2 + t_3) \cdots (-t_2 + t_m) \prod_{2 \leq j \leq k \leq m} (t_j + t_k)} \\ &= \frac{\pm e^{i(\lambda_2 - \lambda_1)t_2} e^{-\varepsilon_2 |t_2|^2}}{2t_2} \mathcal{I}(\lambda_3, \dots, \lambda_m, t_3, \dots, t_m; c). \end{aligned}$$

When we then do the t_2 integral and let $\varepsilon_2 \rightarrow 0^+$, we get zero. Indeed,

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}-i} \frac{e^{i(\lambda_2 - \lambda_1)t_2} e^{-\varepsilon_2 |t_2|^2}}{2t_2} dt_2 = 0,$$

since $\lambda_2 - \lambda_1 < 0$ and the pole at $t_2 = 0$ is above the contour. Similarly, when we compute the residue of the pole $t_1 = -t_j$, $j > 2$, and then perform the t_j integration, we also get zero. Hence we can ignore the residues of the poles $t_1 = -t_j$.

Applying (61) recursively, the integral with $p > 0$ can be reduced to the case with all λ 's negative:

$$(62) \quad \mathcal{I}(\lambda; c) = (-1)^{(m-1)+(m-2)+\dots+(m-p)} (\pi i)^p \mathcal{I}(\lambda_{p+1}, \dots, \lambda_m; c).$$

We now treat the case $p = 0$ (i.e., $0 > \lambda_1 > \dots > \lambda_m$). This time, we do the t_m contour integral first. We close it in the lower half plane, picking up the residue at $t_m = -ic - \sum_{1 \leq k < m} t_k$. This residue is

$$(63) \quad \mathcal{R}(\lambda_1, \dots, \lambda_{m-1}, t_1, \dots, t_{m-1}; c) := \frac{\Delta(t_1, \dots, t_{m-1}) \prod_{k < m} (ic + \sum_{l < m} t_l + t_k) e^{c\lambda_m + i \sum_j (\lambda_j - \lambda_m) t_j}}{2(-ic - \sum_{l < m} t_l) \prod_{1 \leq j \leq k \leq m-1} (t_j + t_k) \prod_{k < m} (-ic - \sum_{l < m, l \neq k} t_l)}.$$

(To simplify the discussion, we set $\varepsilon = 0$, and regard the integrals as distributions, as above.) Next we perform the t_1 integration. Since λ_m is the most negative eigenvalue, we close the contour in the upper half plane. The terms in the denominator with ic all give poles in the lower half plane, so can be ignored. And, the poles $t_1 = -t_j$ will be ignorable, by the same type of reasoning we saw earlier. Indeed, after computing the residue at $t_1 = -t_j$ we find that t_j appears in the exponent as $e^{i(\lambda_j - \lambda_1)t_j}$ with $\lambda_j - \lambda_1 < 0$ and the only factor of the denominator with a zero below the contour is $ic + t_2 + \dots + t_m$; but this factor also appears in the numerator and hence the t_j integral gives zero.

This leaves the residue at all $t_j = 0$ with $1 \leq j \leq m - 1$. The residue at $t_1 = 0$ of (63) is

$$\begin{aligned} & \text{Res}_{|_{t_1=0}} \mathcal{R}(\lambda_1, \dots, \lambda_{m-1}, t_1, \dots, t_{m-1}; c) \\ &= \frac{(-1)^{m-1}}{2} \mathcal{R}(\lambda_2, \dots, \lambda_{m-1}, t_2, \dots, t_{m-1}; c). \end{aligned}$$

Continuing recursively, for the case $p = 0$, we obtain (remembering that the t_m pole below the contour contributes negatively):

$$(64) \quad \mathcal{I}(\lambda; c) = (-1)^{m(m-1)/2} (\pi i)^m \left(\frac{-i}{c} \right) e^{c\lambda_m}.$$

Combining (64) (with m replaced by $m - p$) and (62), we obtain the formula of the proposition. q.e.d.

4.3. Dimensional dependence: a conjecture. In this article, we are studying the N -dependence of the critical point density and number of critical points, but the growth rate of $\mathcal{N}_{N,q}^{\text{crit}}$ (and the growth rate of the total number of critical points) as the dimension $m \rightarrow \infty$ is of considerable interest as well. In the vacuum statistics problem in string/M theory, one is interested in the total number of critical points of certain holomorphic sections (known as flux superpotentials) of a line bundle $\mathcal{L} \rightarrow \mathcal{C}$ over a moduli space \mathcal{C} of complex structures on a Calabi-Yau fourfold $X \times T^2$ where X is a Calabi-Yau threefold and T^2 is an elliptic curve. The relevant dimension is $m = b_3/2 - 1$ where b_3

is the third Betti number of X . As we discuss in some detail in §7.3 of [12], vacuum statistics involve integrals like (7) over certain subspaces of $\text{Sym}(b_3/2 - 1, \mathbb{C})$. The growth rate in b_3 of the number of vacua is important to obtain an order of magnitude of string vacua, since b_3 is rather large for typical Calabi-Yau threefolds. In [12, §7.3], we state a conjectural formula for the growth in b_3 of such integrals which is based on the explicit calculations in this article for $\mathcal{O}(N) \rightarrow \mathbb{CP}^m$ for fixed N .

We recall that by Proposition 4.1, the expected number of critical points on \mathbb{CP}^m of minimum Morse index is of the form $\kappa_m \frac{(N-1)^{m+1}}{(m+2)^{N-2}}$. Our Maple computations given in Appendix 1 show that $\kappa_m = 2(m+1)$, and furthermore that the leading terms of the expansion are monotonically decreasing in the Morse index q , for each $m \leq 6$; in particular, on average there are more critical points on \mathbb{CP}^m of Morse index m than there are of each index $q > m$ (at least for $m \leq 6$). Recall that by our universality results (Theorems 1.1–1.2 and Corollary 1.4), the leading coefficient in the N -expansion of $\mathcal{N}_{N,q}(\mathbb{CP}^m)$ is the universal coefficient $\frac{\pi^m}{m!} b_{0q}(m)$. Thus we make the following conjecture based on our Maple computations:

Conjecture 4.4. Let $n_q(m) := \frac{\pi^m}{m!} b_{0q}(m)$ denote the leading coefficient in the expansion of $\mathcal{N}_{N,q,h}^{\text{crit}}$ from Corollary 1.4, and let $n(m) = \sum_{q=m}^{2m} n_q(m) = \frac{\pi^m}{m!} b_0(m)$, so that

$$\mathcal{N}_{N,q,h}^{\text{crit}} \sim n_q(m) c_1(L)^m N^m, \quad \mathcal{N}_{N,h}^{\text{crit}} \sim n(m) c_1(L)^m N^m.$$

Then

$$n_m(m) = 2 \frac{m+1}{m+2} > n_{m+1}(m) > \dots > n_{2m}(m),$$

and hence

$$2 \frac{m+1}{m+2} < n(m) < 2 \frac{(m+1)^2}{m+2}.$$

Equivalently, $\mathcal{N}_{N,h}^{\text{crit}} \sim n(m) \deg L^N$, where $\deg L^N$ is the number of simultaneous zeros of m generic sections of L^N . This conjecture implies that the expected number of critical points of Morse index m grows exponentially in the dimension, a growth rate consistent with quite analogous estimates of vacua in string/M theory [8, 12] and of metastable states of spin glasses [14].

5. Asymptotics of the expected number of critical points

In this section, we compute the asymptotics of the expected density and number of critical points of sections of powers L^N of a positive holomorphic line bundle. In particular, we prove Theorems 1.1, 1.3, and 1.2 as well as Corollary 1.4.

5.1. Proof of Theorems 1.1–1.2. We begin with some further background on the Szegő kernel.

5.1.1. Szegő kernel asymptotics. We first use the asymptotic expansion of the Szegő kernel to show that $\mathcal{K}_N^{\text{crit}}(z)$ has an expansion of the type given in Theorems 1.1–1.2. It is evident from Theorem 2.1 and Theorem 2.3, respectively, and from formulas (27)–(30) for A_N and Λ_N that the asymptotics of the critical-point densities $\mathcal{K}_N^{\text{crit}}(z)$ and $\mathcal{K}_{N,q}^{\text{crit}}(z)$, respectively, can be determined by canonical algebraic operations on the asymptotics of the following derivatives of the Szegő kernel ($1 \leq j \leq m, 1 \leq j' \leq q \leq m, 1 \leq j' \leq q' \leq m$)

- $\nabla_{z_j} \Pi_N(z, w)|_{z=w}$;
- $\nabla_{z_j} \nabla_{\bar{w}_{j'}} \Pi_N(z, w)|_{z=w}$;
- $\nabla_{z_q} \nabla_{z_j} \Pi_N(z, w)|_{z=w}$ and $\nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi_N(z, w)|_{z=w}$;
- $\nabla_{z_j} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi_N(z, w)|_{z=w}$;
- $\nabla_{z_q} \nabla_{z_j} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi_N(z, w)|_{z=w}$

(Here we write $\Pi(z, w) = \Pi(z, 0; w, 0)$.) We can obtain their asymptotics by differentiating the following Tian-Yau-Zelditch expansion:

Theorem 5.1 ([28, 23, 26]). *Let $(L, h) \rightarrow M$ be a positive Hermitian holomorphic line bundle over a compact complex manifold M of dimension m with Kähler form $\omega_h = \frac{i}{2} \Theta_h$. Then there is a complete asymptotic expansion:*

$$(65) \quad \Pi_N(z, z) \sim \frac{N^m}{\pi^m} [1 + a_1(z)N^{-1} + a_2(z)N^{-2} + \dots] ,$$

for certain smooth coefficients $a_j(z)$.

To apply (65) to the differentiated Szegő kernel, we use (43)–(44). By a change of frame in L , we can assume that K and its holomorphic derivatives up to any fixed order, as well as the anti-holomorphic derivatives, vanish at z_0 . Writing $\partial_j = \partial/\partial z_j$, we then have:

$$(66) \quad \begin{aligned} \nabla_{z_j} \Pi_N(z_0, z_0) &= \frac{\partial F_N}{\partial z_j}(z_0, \bar{z}_0) = \partial_j F_N(z, \bar{z})|_{z_0} \\ &= \partial_j \left[e^{NK(z)} \Pi_N(z, z) \right] \Big|_{z_0} , \\ \nabla_{z_j} \nabla_{\bar{w}_{j'}} \Pi_N(z_0, z_0) &= \frac{\partial^2 F_N}{\partial z_j \partial \bar{w}_{j'}}(z_0, \bar{z}_0) = \partial_j \bar{\partial}_{j'} F_N(z, \bar{z})|_{z_0} \\ &= \partial_j \bar{\partial}_{j'} \left[e^{NK(z)} \Pi_N(z, z) \right] \Big|_{z_0} , \\ &\vdots \\ \nabla_{z_j} \nabla_{z_q} \nabla_{\bar{w}_{j'}} \nabla_{\bar{w}_{q'}} \Pi_N(z_0, z_0) &= \partial_j \partial_q \bar{\partial}_{j'} \bar{\partial}_{q'} \left[e^{NK(z)} \Pi_N(z, z) \right] \Big|_{z_0} . \end{aligned}$$

Here, we used the fact that $F_N(z, \bar{w})$ is holomorphic in z and anti-holomorphic in w . (In these expressions, we have no $\nabla_{\bar{z}_k}$ or ∇_{w_j} derivatives of $\Pi_N(z, w)$.)

It follows by substituting (65) into (66) that the components of A and Λ have asymptotic expansions in powers of N , and hence by Theorem 2.1, resp. Theorem 2.3, that $\mathcal{K}_N^{\text{crit}}(z)$, resp. $\mathcal{K}_{N,q}^{\text{crit}}(z)$, does. Next we study the coefficients b_0, b_1, b_2 of the expansion of $\mathcal{K}_N^{\text{crit}}(z)$.

5.1.2. The first three terms of the expansion. Integrating the density of critical points, we find that the expected total number of critical points has the expansion

$$N^{-m} \mathcal{N}_{N,h}^{\text{crit}} = \frac{\pi^m}{m!} b_0 c_1(L)^m + N^{-1} \int_M b_1 dV_h + N^{-2} \int_M b_2 dV_h + O(N^{-3}).$$

The leading order term is universal.

We will use Theorem 2.1 and the following result of Z. Lu [19] to calculate the coefficients in these expansions:

Theorem 5.2 ([19]). *With the notation as in Theorem 5.1, each coefficient $a_j(z)$ is a polynomial of the curvature and its covariant derivatives at x . In particular,*

$$\begin{cases} a_1 = \frac{1}{2}\rho \\ a_2 = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|Ric|^2 + 3\rho^2) \end{cases}$$

where R, Ric and ρ denotes the curvature tensor, the Ricci curvature and the scalar curvature of ω_h , respectively, and Δ denotes the Laplace operator of (M, ω_h) .

We now calculate A_N and Λ_N to two orders. The key point is to calculate the mixed derivatives of Π_N on the diagonal. It is convenient to do the calculation in Kähler normal coordinates about a point z_0 in M .

It is well known that in terms of Kähler normal coordinates $\{z_j\}$, the Kähler potential K has the expansion:

$$(67) \quad K(z, \bar{z}) = \|z\|^2 - \frac{1}{4} \sum R_{j\bar{k}p\bar{q}}(z_0) z_j \bar{z}_k z_p \bar{z}_q + O(\|z\|^5).$$

(In general, K contains a pluriharmonic term $f(z) + \overline{f(z)}$, but a change of frame for L eliminates that term up to fourth order.)

We further use the notation $K_j = \partial_j K$, $K_{\bar{j}} = \bar{\partial}_j K$. We first claim that

$$(68) \quad A = NI + a_1 I + N^{-1} \{a_2 I + (\partial_j \bar{\partial}_{\bar{j}} a_1)\} + \dots$$

Indeed, by (66),

$$\begin{aligned} \partial_j \left[e^{NK(z)} \Pi_N(z, z) \right] &= e^{NK} \left[NK_j(1 + a_1 N^{-1} + a_2 N^{-2}) \right. \\ &\quad \left. + \partial_j a_1 N^{-1} + \partial_j a_2 N^{-2} + \dots \right] \\ \partial_j \bar{\partial}_{j'} \left[e^{NK(z)} \Pi_N(z, z) \right] &= e^{NK} \left[N^2 K_j K_{\bar{j}'}(1 + a_1 N^{-1} + a_2 N^{-2}) \right. \\ &\quad \left. + K_{\bar{j}'} \partial_j a_1 + K_{\bar{j}'} \partial_j a_2 N^{-1} \right. \\ &\quad \left. + NK_{j\bar{j}'}(1 + a_1 N^{-1} + a_2 N^{-2}) + K_j \bar{\partial}_{j'} a_1 \right. \\ &\quad \left. + K_j \bar{\partial}_{j'} a_2 N^{-1} + \partial_j \bar{\partial}_{j'} a_1 N^{-1} \dots \right]. \end{aligned}$$

Evaluating at z_0 using (67), we then obtain (68).

We now compute the expansion of Λ . Continuing the above computation,

$$\begin{aligned} &\partial_j \bar{\partial}_{j'} \bar{\partial}_{q'} \left[e^{NK(z)} \Pi_N(z, z) \right] \\ &= e^{NK} \left[N^2 (K_{j\bar{q}'} K_{\bar{j}'} + K_{j\bar{j}'} K_{\bar{q}'}) (1 + a_1 N^{-1} + a_2 N^{-2}) \right. \\ &\quad \left. + K_{\bar{j}'} \bar{\partial}_{q'} a_1 + K_{\bar{q}'} \partial_j \bar{\partial}_{j'} a_1 + K_{j\bar{j}'} \bar{\partial}_{q'} a_1 + K_{j\bar{q}'} \bar{\partial}_{j'} a_1 \right. \\ &\quad \left. + NK_{j\bar{j}'} K_{\bar{q}'} (1 + a_1 N^{-1}) \dots \right] + \text{unimportant terms}. \end{aligned}$$

(The ‘unimportant terms’ are those which vanish at z_0 and whose holomorphic derivatives also vanish at z_0 .) We have

$$(69) \quad \begin{aligned} B(z_0) &= \left[(\nabla_{z_j} \nabla_{\bar{w}_{j'}} \nabla_{\bar{w}_{q'}} \Pi_N(z_0, z_0)) \quad (N \nabla_{z_j} \Pi_N(z_0, z_0)) \right] \\ &= \left[(\delta_{j j'} \bar{\partial}_{q'} a_1 + \delta_{j q'} \bar{\partial}_{j'} a_1) \quad (\delta_j a_1) \right] + O(N^{-1}). \end{aligned}$$

Differentiating again and evaluating at z_0 using (67), we obtain

$$\begin{aligned} &\partial_j \bar{\partial}_{j'} \bar{\partial}_q \bar{\partial}_{q'} \left[e^{NK(z)} \Pi_N(z, z) \right] \Big|_{z_0} \\ &= \left[N^2 (\delta_{j j'} \delta_{q q'} + \delta_{j q'} \delta_{q j'}) (1 + a_1 N^{-1} + a_2 N^{-2}) \right. \\ &\quad \left. + \delta_{j j'} \partial_q \bar{\partial}_{q'} a_1 + \delta_{q q'} \partial_j \bar{\partial}_{j'} a_1 + \delta_{j q'} \partial_q \bar{\partial}_{j'} a_1 + \delta_{q j'} \partial_q \bar{\partial}_{j'} a_1 \right. \\ &\quad \left. + NK_{j\bar{j}'} K_{q\bar{q}'} (1 + a_1 N^{-1}) \dots \right] \Big|_{z_0}. \end{aligned}$$

Noting that $K_{j\bar{j}'} K_{q\bar{q}'} \Big|_{z_0} = -R_{j\bar{j}'} K_{q\bar{q}'}(z_0)$, and recalling that $\Lambda = C - B^* A^{-1} B$, where

$$C = \begin{bmatrix} (\nabla_{z_q} \nabla_{z_j} \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi_N) & (N \nabla_{z_q} \nabla_{z_j} \Pi_N) \\ (N \nabla_{\bar{w}_{q'}} \nabla_{\bar{w}_{j'}} \Pi_N) & N^2 \Pi_N \end{bmatrix},$$

we obtain

$$\Lambda(z_0) = N^2 \Lambda_0^{\frac{1}{2}} (I + N^{-1} \Lambda_{-1} + N^{-2} \Lambda_{-2} + \dots) \Lambda_0^{\frac{1}{2}},$$

with

$$(70) \quad \Lambda_0 = \begin{pmatrix} 2\hat{I} & 0 \\ 0 & 1 \end{pmatrix},$$

$$(71) \quad \Lambda_{-1} = \begin{pmatrix} a_1\hat{I} - \frac{1}{2}(R_{jj'\bar{q}\bar{q}'} & 0 \\ 0 & a_1 \end{pmatrix},$$

$$(72) \quad \Lambda_{-2} = \begin{pmatrix} a_2\hat{I} + P - \frac{a_1}{2}(R_{jj'\bar{q}\bar{q}'} & \frac{1}{\sqrt{2}}(\partial_j\partial_q a_1) \\ \frac{1}{\sqrt{2}}(\bar{\partial}_j\bar{\partial}_q a_1) & a_2 \end{pmatrix},$$

where \hat{I} is the identity operator on $\text{Sym}(m, \mathbb{C})$, and

$$P = \frac{1}{2}(\delta_{jj'}\partial_q\bar{\partial}_{q'}a_1 + \delta_{qq'}\partial_j\bar{\partial}_{j'}a_1 + \delta_{jq'}\partial_q\bar{\partial}_{j'}a_1 + \delta_{qj'}\partial_q\bar{\partial}_{j'}a_1).$$

To prove Theorem 1.1, we want the asymptotics of

$$\begin{aligned} \mathcal{K}_N^{\text{crit}}(z_0) &= \frac{\pi^{-(\frac{m+2}{2})} N^m}{\det A_N \det \Lambda_N} \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} |\det(HH^* - |x|^2 I)| \\ &\quad \cdot e^{-\langle \Lambda_N(z_0)^{-1}(H, x), (H, x) \rangle} dH dx. \end{aligned}$$

Making the change of variables $H \mapsto \sqrt{2}NH$, $x \mapsto Nx$, the integral is transformed to

$$(73) \quad \mathcal{K}_N^{\text{crit}}(z_0) = \frac{\pi^{-(\frac{m+2}{2})} N^m}{\det \tilde{A} \det \tilde{\Lambda}} \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} |\det(2HH^* - |x|^2 I)| \\ \cdot e^{-\langle \tilde{\Lambda}^{-1}(H, x), (H, x) \rangle} dH dx,$$

where

$$(74) \quad \begin{aligned} \tilde{A} &= N^{-2}A_N(z_0), \\ \tilde{\Lambda} &= N^{-2}\Lambda_0^{-\frac{1}{2}}\Lambda_N(z_0)\Lambda_0^{-\frac{1}{2}} = (I + N^{-1}\Lambda_{-1} + N^{-2}\Lambda_{-2} + \dots). \end{aligned}$$

Next we observe that

$$\tilde{\Lambda}^{-1} = I - \frac{1}{N}\Lambda_{-1} + \frac{1}{N^2}[-\Lambda_{-2} + \Lambda_{-1}^2] + \dots,$$

hence

$$\begin{aligned} e^{-\langle \tilde{\Lambda}^{-1}H, H \rangle} &\sim e^{-\langle H, H \rangle} e^{\langle [\frac{1}{N}\Lambda_{-1} + \frac{1}{N^2}(\Lambda_{-2} - \Lambda_{-1}^2)]H, H \rangle} \\ &= e^{-\langle H, H \rangle} \left\{ 1 + \frac{1}{N}\langle \Lambda_{-1}H, H \rangle \right. \\ &\quad \left. + \frac{1}{N^2} \left[\langle \Lambda_{-2}H, H \rangle + \frac{1}{2}\langle \Lambda_{-1}H, H \rangle^2 - \langle \Lambda_{-1}^2H, H \rangle \right] \right\}. \end{aligned}$$

Furthermore

$$\det \tilde{\Lambda}^{-1} = 1 - (\text{Tr} \Lambda_{-1})N^{-1} + \left[\frac{1}{2}\text{Tr}(\Lambda_{-1}^2) + \frac{1}{2}(\text{Tr} \Lambda_{-1})^2 - \text{Tr} \Lambda_{-2} \right] N^{-2} \dots,$$

and similarly for $\det A^{-1}$. Altogether, we obtain:

$$\begin{aligned} & \mathcal{K}_N^{\text{crit}}(z) \\ & \sim \pi^{-\binom{m+2}{2}} N^m \left\{ 1 + \frac{1}{N} (-\text{Tr} A_{-1} - \text{Tr} \Lambda_{-1}) + \frac{1}{N^2} \left[\frac{1}{2} \text{Tr}(\Lambda_{-1}^2) \right. \right. \\ & \quad \left. \left. - \text{Tr} \Lambda_{-2} + \frac{1}{2} \text{Tr}(A_{-1}^2) - \text{Tr} A_{-2} + \frac{1}{2} (\text{Tr} A_{-1} + \text{Tr} \Lambda_{-1})^2 \right] \right\} \\ & \quad \cdot \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} |\det(2HH^* - |x|^2 I)| e^{-\langle H, H \rangle} \left\{ 1 + \frac{1}{N} \langle \Lambda_{-1} H, H \rangle \right. \\ & \quad \left. + \frac{1}{N^2} \left[\langle \Lambda_{-2} H, H \rangle + \frac{1}{2} \langle \Lambda_{-1} H, H \rangle^2 - \langle \Lambda_{-1}^2 H, H \rangle \right] \right\} dH dx. \end{aligned}$$

Expanding, we obtain

$$(75) \quad \mathcal{K}_N^{\text{crit}}(z) \sim b_0 N^m + b_1 N^{m-1} + b_2 N^{m-2} + \dots,$$

$$b_0 = \int d\mu,$$

$$b_1 = \int [\langle \Lambda_{-1} H, H \rangle - \text{Tr} A_{-1} - \text{Tr} \Lambda_{-1}] d\mu,$$

$$\begin{aligned} b_2 = \int & \left[\frac{1}{2} \text{Tr}(\Lambda_{-1}^2) - \text{Tr} \Lambda_{-2} + \frac{1}{2} \text{Tr}(A_{-1}^2) - \text{Tr} A_{-2} \right. \\ & \left. + \frac{1}{2} (\text{Tr} A_{-1} + \text{Tr} \Lambda_{-1})^2 - (\text{Tr} A_{-1} + \text{Tr} \Lambda_{-1}) \langle \Lambda_{-1} H, H \rangle \right. \\ & \left. + \langle (\Lambda_{-2} - \Lambda_{-1}^2) H, H \rangle + \frac{1}{2} \langle \Lambda_{-1} H, H \rangle^2 \right] d\mu, \end{aligned}$$

where

$$(76) \quad d\mu = \pi^{-\binom{m+2}{2}} |\det(2HH^* - |x|^2 I)| e^{-\langle H, H \rangle} dH dx.$$

Recalling (68) and (70)–(72), we see that b_1 is of the form

$$b_1 = \sum c_{j\bar{j}'q\bar{q}'} R_{j\bar{j}'q\bar{q}'},$$

where $c_{j\bar{j}'q\bar{q}'}$ is universal. Since b_1 is also invariant under the unitary group, we must have

$$(77) \quad b_1 = \beta_1 \rho,$$

where β_1 is a universal constant (depending only on the dimension m of M). Similarly, b_2 is of the form

$$b_2 = Q(R, R) + \gamma_0 \Delta \rho,$$

where $Q(R, R)$ is a universal quadratic form in the curvature tensor R . But b_2 is also $U(m)$ -invariant and hence is a curvature invariant (of

order 2). Thus,

$$(78) \quad b_2 = \gamma_0 \Delta \rho + \gamma_1 \rho^2 + \gamma_2 |R|^2 + \gamma_3 |Ric|^2,$$

where the γ_k are universal constants depending only on m , which completes the proof of Theorem 1.1.

The proof of Theorem 1.2 is exactly as above, except we integrate over $\mathbf{S}_{m,q-m}$ instead of $\text{Sym}(\mathbb{C}, m) \times \mathbb{C}$ in the computation of the expansion of $\mathcal{K}_{N,q}^{\text{crit}}$. q.e.d.

5.2. Number of critical points: proof of Corollary 1.4. The coefficient b_{1q} is of the form

$$(79) \quad b_{1q} = \beta_{1q} \rho,$$

by the above argument (or by the fact that ρ is the only curvature invariant of order 1). Furthermore, it is well known (see, e.g., [18, pp. 112–113]) that for any Kähler metric ω on M , we have

$$(80) \quad \begin{aligned} (\rho^2 - |\text{Ric}|^2)\Omega &= c_1(M, \omega)^2 \wedge \omega^{m-2}, \\ (|\text{Ric}|^2 - |R|^2)\Omega &= [c_1(M, \omega)^2 - 2c_2(M, \omega)] \wedge \omega^{m-2}, \end{aligned}$$

where $\Omega = \frac{1}{4\pi^2 m(m-1)} \omega^m$. Therefore

$$(81) \quad \begin{aligned} b_{2q} &= \gamma_{0q} \Delta \rho + (\gamma_{1q} + \gamma_{2q} + \gamma_{3q}) \rho^2 + \text{const.} \frac{c_1(h)^2 \wedge \omega_h^{m-2}}{\omega_h^m} \\ &+ \text{const.} \frac{c_2(h) \wedge \omega_h^{m-2}}{\omega_h^m}, \end{aligned}$$

where we now write $c_j(h) = c_j(M, \omega_h)$ for the j -th Chern form of the Kähler metric $\omega_h = -\frac{i}{2} \partial \bar{\partial} \log h$.

Integrating (77) and (81), noting that $\int \Delta \rho dV_h = \frac{2}{m!} \int \partial \bar{\partial} \rho \wedge \omega_h^{m-1} = 0$, we obtain the asymptotic expansion of Corollary 1.4 with $\beta_{2q} = \gamma_{1q} + \gamma_{2q} + \gamma_{3q}$. q.e.d.

5.3. Asymptotic expansions on Riemann surfaces: proof of Theorem 1.3. On a compact Riemann surface C of genus g , Corollary 1.4 says that $\mathcal{N}_{N,q,h}^{\text{crit}}$ has a universal expansion of the form

$$\mathcal{N}_{N,q,h}^{\text{crit}} \sim \pi b_{0q} c_1(L) N + \pi \beta_{1q} (2 - 2g) + \beta_{2q} \int_C \rho^2 \omega_h N^{-1} + \dots,$$

for $q = 1, 2$. There are several ways to compute the constants. A quick way to find b_0, β_1, β_2 , is to consider the case of \mathbb{CP}^1 with the Fubini-Study metric on $L = \mathcal{O}(1)$. By an elementary computation in [11] (or

by §4), we showed that for this case

$$\begin{aligned} \mathcal{N}_{N,1,h}^{\text{crit}} &= \frac{4(N-1)^2}{3N-2} = \frac{4}{3}N - \frac{16}{9} + \frac{4}{27}N^{-1} \dots \\ &\quad \text{(expected number of saddle points),} \\ \mathcal{N}_{N,2,h}^{\text{crit}} &= \frac{N^2}{3N-2} = \frac{1}{3}N + \frac{2}{9} + \frac{4}{27}N^{-1} \dots \\ &\quad \text{(expected number of local maxima).} \end{aligned}$$

Therefore,

$$\pi b_{01} = \frac{4}{3}, \quad \pi b_{02} = \frac{1}{3}, \quad \pi \beta_{11} = -\frac{8}{9}, \quad \pi \beta_{02} = \frac{1}{9}.$$

To find β_{21}, β_{22} , we note that $\int_{\mathbb{C}P^1} \omega_{\text{FS}} = \pi c_1(L) = \pi$, where ω_{FS} is the Fubini-Study Kähler form on $\mathbb{C}P^1$. Furthermore, since $c_1(\mathbb{C}P^1) = \frac{1}{\pi} \int \rho \omega_{\text{FS}} = 2$, we have $\rho \equiv 2$. (This can be checked directly as follows: the Kähler potential $K = \log(1 + |z|^2) = |z|^2 - \frac{1}{2}|z|^4 + \dots$, where z is the affine coordinate, and hence by (67), $\rho(0) = R_{1\bar{1}1\bar{1}}(0) = 2$.) Hence, $\int \rho^2 \omega_{\text{FS}} = 4\pi$, and therefore

$$\beta_{21} = \beta_{22} = \frac{1}{27\pi},$$

which completes the proof of Theorem 1.3. q.e.d.

6. Proof of Theorem 1.6: evaluating the coefficient $\beta_{2q}(m)$

We have already shown that

- $\int_M b_{1q} dV_h$ is topological;
- $\int_M b_{2q} dV_h$ is the sum of a topological term plus a positive multiple of $\int_M \rho_h^2 dV_h$.

To complete the proof of Theorem 1.6 and show that the metric with asymptotically minimal $\mathcal{N}_{N,q,h}^{\text{crit}}$ (and $\mathcal{N}_{N,h}^{\text{crit}}$) is the one for which ω_h has minimal \mathcal{L}^2 norm of the scalar curvature, we must show that the β_{2q} are positive.

The proof consists of finding an integral formula for β_{2q} and then transforming it to one that is amenable to (computer) evaluation. We first summarize the key results:

Lemma 6.1. *In all dimensions,*

$$(82) \quad \beta_{2q}(m) = \frac{1}{4\pi^{\binom{m+2}{2}}} \int_{\mathbf{S}_{m,q-m}} \gamma(H) |\det(2HH^* - |x|^2 I)| \cdot e^{-\langle(H,x),(H,x)\rangle} dH dx,$$

where

$$\gamma(H) = \frac{1}{2} |H_{11}|^4 - 2|H_{11}|^2 + 1,$$

and $\mathbf{S}_{m,q-m}$ is given by (9).

After a sequence of manipulations as in the proof of Lemma 3.1, the integral (82) will be rewritten in the following form:

Lemma 6.2.

$$\beta_{2q}(m) = \frac{(-i)^{m(m-1)/2}}{4 \pi^{2m} \prod_{j=1}^{m-1} j!} \int_{Y_{2m-q}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Delta(\lambda) \Delta(\xi) \cdot \prod_{j=1}^m |\lambda_j| e^{i\langle \lambda, \xi \rangle} \mathcal{I}(\lambda, \xi) d\xi_1 \cdots d\xi_m d\lambda,$$

where

$$(83) \quad \mathcal{I}(\lambda, \xi) = \frac{F(D(\lambda)) + \left[\frac{4 \sum_{j=1}^m \lambda_j}{m(m+1)(m+3)} - \frac{2}{m+1} \right] \frac{1}{(1 - \frac{i}{2} \sum_j \xi_j)} + \frac{2}{(m+1)(m+3)(1 - \frac{i}{2} \sum_j \xi_j)^2}}{\left(1 - \frac{i}{2} \sum_j \xi_j \right) \prod_{j < k} \left[1 + \frac{i}{2} (\xi_j + \xi_k) \right]}.$$

Here, $D(\lambda)$ is the diagonal matrix with diagonal entries $\lambda = (\lambda_1, \dots, \lambda_m)$, $\Delta(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j)$ is the Vandermonde determinant and

$$F(P) = 1 - \frac{4 \operatorname{Tr} P}{m(m+1)} + \frac{4(\operatorname{Tr} P)^2 + 8 \operatorname{Tr}(P^2)}{m(m+1)(m+2)(m+3)},$$

for (Hermitian) $m \times m$ matrices P . The iterated $d\xi_j$ integrals are defined in the distribution sense.

The final step is the evaluation of $\beta_{2q}(m)$. Having simplified the integral as far as we could, we complete the computation for the cases $m \leq 5$ using Maple and find that it is positive for these cases. The resulting values of the constants $\beta_{2q}(m)$, $m \leq 4$, are given in §6.4.

6.1. Proof of Lemma 6.1. We use the case of $M = \mathbb{C}\mathbb{P}^1 \times E^{m-1}$ where E is an elliptic curve, and L is the product of degree 1 line bundles on the factors (with the Fubini-Study metric on $\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(1)$ and the flat metric on E). (The manifold M is a homogeneous space with respect to $SU(2) \times T^{2m-2}$, so the critical point density is invariant and hence constant.)

Since $c_1(h)^2 = c_2(h) = 0$, it follows from (81) that the coefficient b_{2q} of the expansion $N^{-m} \mathcal{K}_N^{\text{crit}}(z) = b_{0q} + b_{1q} N^{-1} + b_{2q} N^{-2} + O(N^{-3})$ is given by $b_{2q} = \beta_{2q} \rho^2$, and hence

$$(84) \quad \beta_{2q} = \frac{1}{\rho^2} b_{2q} = \frac{1}{4} b_{2q}.$$

The Szegő kernel for (M, L) is the product of the Szegő kernels on $\mathbb{C}\mathbb{P}^1$ and E^{m-1} . Since the universal cover of E^{m-1} is \mathbb{C}^{m-1} , the Szegő

kernel on E^{m-1} is given by the Heisenberg Szegő kernel on \mathbb{C}^{m-1} (see [3, §1.3.2]) modulo an $O(N^{-\infty})$ term, and we have:

$$\begin{aligned} \Pi_{CP^1 \times E^{m-1}}(z, w) &= \frac{(N+1)N^{m-1}}{\pi^m} (1+z_1\bar{w}_1)^N e^{N(z_2\bar{w}_2+\dots+z_m\bar{w}_m)} \\ &\quad \cdot e_L(z) \otimes \overline{e_L(w)} + O(N^{-\infty}). \end{aligned}$$

As in §4, we consider the normalized Szegő kernel

$$(85) \quad \begin{aligned} \tilde{\Pi}_N(z, w) &:= (1+z_1\bar{w}_1)^N e^{Nz'\bar{w}'}, \\ z' &= (z_2, \dots, z_m), \quad w' = (w_2, \dots, w_m). \end{aligned}$$

We have:

$$\begin{aligned} \frac{\partial \tilde{\Pi}_N}{\partial z_1} &= N(1+z_1\bar{w}_1)^{N-1} e^{Nz'\bar{w}'} \bar{w}_1, \\ \frac{\partial \tilde{\Pi}_N}{\partial z_\alpha} &= N(1+z_1\bar{w}_1)^N e^{Nz'\bar{w}'} \bar{w}_\alpha, \\ \frac{\partial^2 \tilde{\Pi}_N}{\partial z_1 \partial \bar{w}_1} &= \{N(1+z_1\bar{w}_1)^{N-1} + N(N-1)(1+z_1\bar{w}_1)^{N-2} z_1\bar{w}_1\} e^{Nz'\bar{w}'}, \\ \frac{\partial^2 \tilde{\Pi}_N}{\partial z_\alpha \partial \bar{w}_{\alpha'}} &= \{N\delta_{\alpha\alpha'} + N^2 z_{\alpha'} \bar{w}_\alpha\} (1+z_1\bar{w}_1)^N e^{Nz'\bar{w}'}, \\ \frac{\partial^2 \tilde{\Pi}_N}{\partial z_1 \partial \bar{w}_\alpha} &= N^2 (1+z_1\bar{w}_1)^{N-1} e^{Nz_\alpha \bar{w}_\alpha} z_\alpha \bar{w}_1, \\ \frac{\partial^2 \tilde{\Pi}_N}{\partial z_\alpha \partial \bar{w}_1} &= \{N^2 (1+z_1\bar{w}_1)^{N-1} z_1 \bar{w}_\alpha\} e^{Nz'\bar{w}'}, \\ &\quad 2 \leq \alpha, \alpha' \leq m. \end{aligned}$$

It suffices to compute the density at 0. From the above, we have:

$$(86) \quad \frac{\partial^4 \tilde{\Pi}_N}{\partial z_j \partial z_q \partial \bar{w}_{j'} \partial \bar{w}_{q'}}(0, 0) = \begin{cases} 2N(N-1), & j = q = j' = q' = 1 \\ 2N^2, & j = q = j' = q' > 1 \\ N^2, & j = j' \neq q' = q \end{cases}.$$

Recalling (28)–(30), we then have:

$$(87) \quad \tilde{A}_N(0) = \left(\frac{\partial^2 \tilde{\Pi}_N}{\partial z_j \partial \bar{w}_{j'}}(0, 0) \right) = NI,$$

$$(88) \quad \tilde{B}_N(0) = \left[\left(\tau_{jq} \frac{\partial^3 \tilde{\Pi}_N}{\partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}}(0, 0) \right) \left(N \frac{\partial \tilde{\Pi}_N}{\partial z_j}(0, 0) \right) \right] = 0,$$

$$(89) \quad \tilde{C}_N(0) = \begin{bmatrix} \left(\tau_{jq} \tau_{j'q'} \frac{\partial^4 \tilde{\Pi}_N}{\partial z_q \partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}}(0, 0) \right) & \left(\tau_{jq} N \frac{\partial^2 \tilde{\Pi}_N}{\partial z_j \partial z_q}(0, 0) \right) \\ \left(\tau_{j'q'} N \frac{\partial^2 \tilde{\Pi}_N}{\partial \bar{w}_{q'} \partial \bar{w}_{j'}}(0, 0) \right) & N^2 \tilde{\Pi}_N(0, 0) \end{bmatrix},$$

$$1 \leq j \leq m, 1 \leq j \leq q \leq m, 1 \leq j' \leq q' \leq m.$$

It follows that

$$(90) \quad \tilde{\Lambda}_N(0) = \tilde{C}_N(0) = D(2N(N-1), \overbrace{2N^2, \dots, 2N^2}^{(m-1)(m+2)/2}, N^2),$$

i.e., the diagonal matrix with diagonal entries $2N(N-1)$, $2N^2$ repeated $(m-1)(m+2)/2$ times, and N^2 .

We want to compute

$$\begin{aligned} \mathcal{K}_{N,q}^{\text{crit}}(0) &= \frac{\pi^{-\binom{m+2}{2}}}{\det \tilde{A}_N(0) \det \tilde{\Lambda}_N(0)} \int_{\mathbf{S}_{m,q-m}} |\det(HH^* - |x|^2 I)| \\ &\quad \cdot e^{-\langle \tilde{\Lambda}_N(0)^{-1}(H,x), (H,x) \rangle} dH dx. \end{aligned}$$

Making the change of variables $H \mapsto \sqrt{2}NH$, $x \mapsto Nx$, the integral is transformed to

$$(91) \quad \mathcal{K}_{N,q}^{\text{crit}}(0) = \frac{\pi^{-\binom{m+2}{2}} N^m}{\det \hat{\Lambda}} \int_{\mathbf{S}_{m,q-m}} |\det(2HH^* - |x|^2 I)| \\ \cdot e^{-\langle \hat{\Lambda}^{-1}(H,x), (H,x) \rangle} dH dx,$$

where

$$\hat{\Lambda} = I - \frac{1}{N}E, \quad E = D(1, 0, \dots, 0).$$

Therefore

$$\begin{aligned}
 & N^{-m} \mathcal{K}_{N,q}^{\text{crit}}(0) \\
 &= \pi^{-\binom{m+2}{2}} \left(1 + \frac{1}{N} + \frac{1}{N^2} + \dots \right) \int_{\mathbf{S}_{m,q-m}} |\det(2HH^* - |x|^2 I)| \\
 &\quad \cdot \exp \left(-\|H\|^2 - |x|^2 - \frac{1}{N} |H_{11}|^2 - \frac{1}{N^2} |H_{11}|^2 - \dots \right) dH dx \\
 &= \left(1 + \frac{1}{N} + \frac{1}{N^2} + \dots \right) \int \exp \left(-\frac{1}{N} |H_{11}|^2 - \frac{1}{N^2} |H_{11}|^2 - \dots \right) d\tilde{\mu} \\
 &= \int \left[1 + \frac{1}{N} (1 - |H_{11}|^2) + \frac{1}{N^2} \left(1 - 2|H_{11}|^2 + \frac{1}{2} |H_{11}|^4 \right) \right] d\tilde{\mu} \\
 &\quad + O \left(\frac{1}{N^3} \right),
 \end{aligned}$$

where

$$d\tilde{\mu} = \pi^{-\binom{m+2}{2}} |\det(2HH^* - |x|^2 I)| e^{-\langle (H,x), (H,x) \rangle} dH dx.$$

Therefore

$$b_{2q} = \int_{\mathbf{S}_{m,q-m}} \left(1 - 2|H_{11}|^2 + \frac{1}{2} |H_{11}|^4 \right) d\tilde{\mu},$$

and the desired formula then follows from (84). q.e.d.

6.2. U(m) symmetries of the integral. As an intermediate step between Lemmas 6.1 and 6.2, we prove:

Lemma 6.3.

$$\begin{aligned}
 (92) \quad \beta_{2q}(m) &= \frac{1}{4 \pi^{\binom{m+2}{2}}} \int_{\mathbf{S}_{m,q-m}} F(HH^*) |\det(2HH^* - |x|^2 I)| \\
 &\quad \cdot e^{-\langle (H,x), (H,x) \rangle} dH dx,
 \end{aligned}$$

where

$$(93) \quad F(P) = 1 - \frac{4 \operatorname{Tr} P}{m(m+1)} + \frac{4(\operatorname{Tr} P)^2 + 8 \operatorname{Tr}(P^2)}{m(m+1)(m+2)(m+3)},$$

for (Hermitian) $m \times m$ matrices P .

Proof. Since the change of variables $H \mapsto gHg^t$ ($g \in \text{U}(m)$) is unitary on $\text{Sym}(m, \mathbb{C})$ (with respect to the Hilbert-Schmidt inner product), we can make this change of variables in (82), and then integrate over $g \in \text{U}(m)$ to obtain

$$\begin{aligned}
 (94) \quad \beta_{2q}(m) &= \frac{1}{4 \pi^{\binom{m+2}{2}}} \int_{\mathbf{S}_{m,q-m}} \left(\int_{\text{U}(m)} \gamma(gHg^t) dg \right) \\
 &\quad \cdot |\det(2HH^* - |x|^2 I)| e^{-\langle (H,x), (H,x) \rangle} dH dx.
 \end{aligned}$$

We now evaluate the integral $\int_{\text{U}(m)} \gamma(gHg^t) dg$.

Claim. For $H \in \text{Sym}(m, \mathbb{C})$,

$$(95) \quad \int_{\text{U}(m)} |(gHg^t)_{11}|^2 dg = \frac{2}{m(m+1)} \text{Tr}(HH^*),$$

$$(96) \quad \int_{\text{U}(m)} |(gHg^t)_{11}|^4 dg = \frac{8(\text{Tr} HH^*)^2 + 16 \text{Tr}(HH^*HH^*)}{m(m+1)(m+2)(m+3)}.$$

To prove the claim, we write $v = (v_1, \dots, v_m) = (g_{11}, \dots, g_{1m})$ so that $(gHg^t)_{11} = vHv^t$, and we replace $\int_{\text{U}(m)} dg$ with $\int_{S^{2m-1}} d\nu(v)$, where $d\nu$ is Haar probability measure on S^{2m-1} . Next we recall that if p is a homogeneous polynomial of degree $2k$ on \mathbb{R}^{2m} ,

$$(97) \quad \int_{S^{2m-1}} p(v) d\nu(v) = \frac{(m-1)!}{(m-1+k)!} \int_{\mathbb{R}^{2m}} p(v) d\gamma(v),$$

$$d\gamma(v) = \frac{1}{\pi^m} e^{-\|v\|^2} dv.$$

We easily see using Wick's formula that

$$\begin{aligned} \int_{\mathbb{C}^m} |vHv^t|^2 d\gamma &= \sum_{j,k,j',k'} H_{jk} \bar{H}_{j'k'} \int_{\mathbb{C}^m} v_j v_k \bar{v}_{j'} \bar{v}_{k'} d\gamma \\ &= \sum_j |H_{jj}|^2 \int_{\mathbb{C}^m} |v_j|^4 d\gamma \\ &\quad + 2 \sum_{j \neq k} |H_{jk}|^2 \int_{\mathbb{C}^m} |v_j|^2 |v_k|^2 d\gamma \\ &= 2 \text{Tr}(HH^*). \end{aligned}$$

Formula (95) then follows from (97) with $k = 2$.

Although the above approach can also be used to verify (96), we find it easier to use invariant theory, since the integral in (96) is a $\text{U}(m)$ -invariant function of $H \in \text{Sym}(m, \mathbb{C})$, under the $\text{U}(m)$ action $H \mapsto gHg^t$. Indeed, it is a $\text{U}(m)$ -invariant Hermitian inner product on the symmetric product $\mathcal{S}^2(\text{Sym}(m, \mathbb{C})) \approx \mathcal{S}^2(\mathcal{S}^2(\mathbb{C}^m))$.

The action of $\text{U}(m)$ on symmetric complex matrices defines a representation equivalent to $\mathcal{S}^2(\mathbb{C}^m)$ where \mathbb{C}^m is the defining representation of $\text{U}(m)$. It is well known from Schur-Weyl duality that $\mathcal{S}^2(\mathbb{C}^m)$ is irreducible. We then consider the $\text{U}(m)$ representation

$$\mathcal{S}^2(\mathcal{S}^2(\mathbb{C}^m)) = \mathbb{C}\{H_1 \otimes H_2 + H_2 \otimes H_1, \quad H_1, H_2 \in \mathcal{S}^2(\mathbb{C}^m)\},$$

with the diagonal action. Henceforth we put

$$H_1 \cdot H_2 := \frac{1}{2}[H_1 \otimes H_2 + H_2 \otimes H_1].$$

We then regard $F(H)$ as the value on $H \otimes H$ of the quadratic form

$$Q(H_1 \cdot H_2) = \int_{\text{U}(m)} |\langle gH_1 g^t \cdot gH_2 g^t e_1 \otimes e_1, e_1 \otimes e_1 \rangle|^2 dg.$$

This defines the Hermitian inner product

$$\begin{aligned} & \langle\langle H_1 \cdot H_2, H_3 \cdot H_4 \rangle\rangle \\ &= \int_{U(m)} \langle gH_1g^t \cdot gH_2g^t e_1 \otimes e_1, e_1 \otimes e_1 \rangle \overline{\langle gH_3g^t \cdot gH_4g^t e_1 \otimes e_1, e_1 \otimes e_1 \rangle} dg. \end{aligned}$$

We next recall that $\mathcal{S}^2(\mathcal{S}^2(\mathbb{C}^m))$ decomposes into a direct sum of two $U(m)$ irreducibles, one corresponding to the Young diagram Y_1 with 1 row of four boxes and one corresponding to the diagram Y_2 with 2 rows each with two boxes. See for instance Proposition 1 of [16]. The Young projectors are, respectively,

$$\begin{cases} P_{Y_1}(H \otimes H)_{i_1 i_2 i_3 i_4} = \sum_{\sigma \in S_4} H_{i_{\sigma(1)} i_{\sigma(2)}} H_{i_{\sigma(3)} i_{\sigma(4)}} \\ P_{Y_2}(H \otimes H)_{i_1 i_2 i_3 i_4} = \sum_{\sigma \in S_2 \times S_2} (-1)^\sigma H_{i_{\sigma(1)} i_{\sigma(2)}} H_{i_{\sigma(3)} i_{\sigma(4)}}. \end{cases}$$

For Y_2 the $S_2 \times S_2$ permutes $1 \iff 3, 2 \iff 4$.

Since an irreducible $U(m)$ representation has (up to scalar multiples) a unique invariant inner product, it follows that

$$\langle\langle \cdot, \cdot \rangle\rangle = c_1 \langle \cdot, \cdot \rangle_{Y_1} + c_2 \langle \cdot, \cdot \rangle_{Y_2},$$

where $\langle \cdot, \cdot \rangle_{Y_j}$ are the invariant inner products

$$\langle A, B \rangle_{Y_j} = \text{Tr } P_{Y_j}(A)B^*$$

for the irreducibles corresponding to the Young diagrams Y_j as above.

We now calculate these inner products on $H \otimes H$. We have

$$\begin{aligned} \|H \otimes H\|_{Y_1}^2 &= \sum_{\sigma \in S_4} \sum_{i_1, i_2, i_3, i_4=1}^m H_{i_{\sigma(1)} i_{\sigma(2)}} H_{i_{\sigma(3)} i_{\sigma(4)}} \bar{H}_{i_1 i_2} \bar{H}_{i_3 i_4}, \\ \|H \otimes H\|_{Y_2}^2 &= \sum_{\sigma \in S_2 \times S_2} \sum_{i_1, i_2, i_3, i_4=1}^m (-1)^\sigma H_{i_{\sigma(1)} i_{\sigma(2)}} H_{i_{\sigma(3)} i_{\sigma(4)}} \bar{H}_{i_1 i_2} \bar{H}_{i_3 i_4}. \end{aligned}$$

It is easy to see that each of these expressions is a linear combination of the two quadratic forms

$$H \otimes H \mapsto \text{Tr}\{[H \otimes H] \circ [H^* \otimes H^*]\}, \quad H \otimes H \mapsto [\text{Tr } H \circ H^*]^2.$$

Hence

$$\int_{U(m)} |(gHg^t)_{11}|^4 dg = c_1 (\text{Tr } HH^*)^2 + c_2 \text{Tr}(HH^*HH^*).$$

To determine the constants c_1, c_2 , it suffices to consider the case where H is diagonal. Let s_1, \dots, s_m denote the eigenvalues of H . Then by

Wick’s formula we obtain

$$\begin{aligned}
 \int_{\mathbb{C}^m} |vHv^t|^4 d\gamma &= \sum_{j,k,j',k'} s_j s_k \bar{s}_{j'} \bar{s}_{k'} \int_{\mathbb{C}^m} v_j^2 v_k^2 \bar{v}_{j'}^2 \bar{v}_{k'}^2 d\gamma \\
 &= \sum_j |s_j|^4 \int_{\mathbb{C}^m} |v_j|^8 d\gamma \\
 &\quad + 2 \sum_{j \neq k} |s_j|^2 |s_k|^2 \int_{\mathbb{C}^m} |v_j|^4 |v_k|^4 d\gamma \\
 &= 4! \sum_j |s_j|^4 + 8 \sum_{j \neq k} |s_j|^2 |s_k|^2 \\
 &= 8 (\text{Tr } HH^*)^2 + 16 \text{Tr } (HH^*)^2.
 \end{aligned}$$

Formula (96) now follows from (97) with $k = 4$.

Having proved the claim, the formula stated in Lemma 6.3 now follows from (94) and Lemma 6.2. q.e.d.

6.3. Proof of Lemma 6.2. We proceed exactly as in the proof of Lemma 3.1. We rewrite the integral (82) as

$$(98) \quad \beta_{2q}(m) = \frac{1}{4 \pi^m (2\pi)^{m^2}} \lim_{\varepsilon' \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon, \varepsilon'},$$

where

$$\begin{aligned}
 (99) \quad \mathcal{I}_{\varepsilon, \varepsilon'} &= \frac{1}{\pi^{d_m}} \int_{\mathcal{H}_m} d\Xi \int_{\mathcal{H}_m(m-q)} dP \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} dH dx \\
 &\quad \cdot F \left(P + \frac{1}{2} |x|^2 I \right) |\det(2P)| e^{i \langle \Xi, P - HH^* + \frac{1}{2} |x|^2 I \rangle} \\
 &\quad \cdot \exp(-\text{Tr} HH^* - |x|^2) \exp(-\varepsilon \text{Tr } \Xi \Xi^* - \varepsilon' \text{Tr } PP^*), \\
 \mathcal{H}_m(m-q) &= \{P \in \mathcal{H}_m : \text{index } P = m - q\}.
 \end{aligned}$$

Recall that $d_m = \dim_{\mathbb{C}}(\text{Sym}(m, \mathbb{C}) \times \mathbb{C}) = \frac{1}{2}(m^2 + m + 2)$. As in §3.1, we note that absolute convergence is guaranteed by the Gaussian factors in each variable (H, x, P, Ξ) . Evaluating $\int e^{i \langle \Xi, P - HH^* + \frac{1}{2} |x|^2 I \rangle} e^{-\varepsilon \text{Tr } \Xi \Xi^*} d\Xi$ first, we obtain a dual Gaussian, which approximates the delta function $\delta_{HH^* - \frac{1}{2} |x|^2 I}(P)$. As $\varepsilon \rightarrow 0$, the dP integral then yields the integrand at $P = HH^* - \frac{1}{2} |x|^2 I$; then letting $\varepsilon' \rightarrow 0$ we obtain the original integral stated in Lemma 6.3.

Continuing as in §3.1, we conjugate P to a diagonal matrix $D(\lambda)$ with $\lambda = (\lambda_1, \dots, \lambda_m)$ by an element $h \in U(m)$ and we replace dP with

$\Delta(\lambda)^2 d\lambda dh$. Recalling (47), we obtain:

$$\begin{aligned} \mathcal{I}_{\varepsilon, \varepsilon'} &= \frac{2^m c'_m}{\pi^{d_m}} \int_{U(m)} dh \int_{\mathcal{H}_m} d\Xi \int_{Y'_{2m-q}} d\lambda \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} dH dx \\ &\cdot \Delta(\lambda)^2 \prod_{j=1}^m |\lambda_j| F(D(\lambda) + \frac{1}{2}|x|^2 I) \\ &\cdot e^{i\langle \Xi, hD(\lambda)h^* + \frac{1}{2}|x|^2 I - HH^* \rangle} e^{-[\text{Tr}HH^* + |x|^2 + \varepsilon \text{Tr}\Xi\Xi^* + \varepsilon' \sum \lambda_j^2]}. \end{aligned}$$

Again using (47) applied this time to Ξ , we obtain:

$$\begin{aligned} \mathcal{I}_{\varepsilon, \varepsilon'} &= \frac{2^m (c'_m)^2}{\pi^{d_m}} \int_{U(m)} dg \int_{U(m)} dh \int_{\mathbb{R}^m} d\xi \int_{Y'_{2m-q}} d\lambda \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} dH dx \\ &\cdot \Delta(\lambda)^2 \Delta(\xi)^2 \prod_{j=1}^m |\lambda_j| F(D(\lambda) + \frac{1}{2}|x|^2 I) \\ &\cdot e^{i\langle gD(\xi)g^*, hD(\lambda)h^* + \frac{1}{2}|x|^2 I - HH^* \rangle} e^{-\text{Tr}HH^* - |x|^2 - \sum(\varepsilon\xi_j^2 + \varepsilon'\lambda_j^2)}. \end{aligned}$$

We then transfer the conjugation by g to the right side of the \langle, \rangle in the first exponent and make the change of variables $h \mapsto gh, H \mapsto gHg^t$ to eliminate g from the integrand:

$$\begin{aligned} \mathcal{I}_{\varepsilon, \varepsilon'} &= \frac{2^m (c'_m)^2}{\pi^{d_m}} \int_{U(m)} dh \int_{\mathbb{R}^m} d\xi \int_{Y'_{2m-q}} d\lambda \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} dH dx \\ &\cdot \Delta(\lambda)^2 \Delta(\xi)^2 \prod_{j=1}^m |\lambda_j| F(D(\lambda) + \frac{1}{2}|x|^2 I) \\ &\cdot e^{i\langle D(\xi), hD(\lambda)h^* + \frac{1}{2}|x|^2 I - HH^* \rangle} e^{-\text{Tr}HH^* - |x|^2 - \sum(\varepsilon\xi_j^2 + \varepsilon'\lambda_j^2)}. \end{aligned}$$

Next we substitute the Itzykson-Zuber-Harish-Chandra integral formula (49) into the above and expand

$$\det[e^{i\xi_j \lambda_k}]_{jk} = \sum_{\sigma \in S_m} (-1)^\sigma e^{i\langle \xi, \sigma(\lambda) \rangle},$$

obtaining a sum of $m!$ integrals. However, by making the change of variables $\lambda' = \sigma(\lambda)$ and noting that $\Delta(\lambda') = (-1)^\sigma \Delta(\lambda)$, we see as

before that the integrals of these terms are equal, and so we obtain

$$\begin{aligned}
 (100) \quad \mathcal{I}_{\varepsilon, \varepsilon'} &= (-i)^{m(m-1)/2} \frac{c_m''}{\pi^{dm}} \int_{\mathbb{R}^m} d\xi \int_{Y'_{2m-q}} d\lambda \int_{\text{Sym}(m, \mathbb{C}) \times \mathbb{C}} dH dx \\
 &\quad \cdot \Delta(\lambda) \Delta(\xi) \prod_{j=1}^m |\lambda_j| e^{i\langle \lambda, \xi \rangle} F \left(D(\lambda) + \frac{1}{2} |x|^2 I \right) \\
 &\quad \cdot \exp \left(i \left\langle D(\xi), \frac{1}{2} |x|^2 I - HH^* \right\rangle - \text{Tr} HH^* - |x|^2 \right) \\
 &\quad \cdot \exp \left(-\varepsilon \sum \xi_j^2 - \varepsilon' \sum \lambda_j^2 \right)
 \end{aligned}$$

where

$$c_m'' = \frac{2^{m^2} \pi^{m(m-1)}}{\prod_{j=1}^m j!}.$$

The phase

$$\begin{aligned}
 (101) \quad \Phi(H, x; \xi) &:= i \left\langle D(\xi), \frac{1}{2} |x|^2 I - HH^* \right\rangle - \text{Tr} HH^* - |x|^2 \\
 &= - \left[\|H\|_{\text{HS}}^2 + i \sum_{j,k=1}^m \xi_j |H_{jk}|^2 + \left(1 - \frac{i}{2} \sum_j \xi_j \right) |x|^2 \right] \\
 &= - \left[\sum_{j \leq k} \left(1 + \frac{i}{2} (\xi_j + \xi_k) \right) |\widehat{H}_{jk}|^2 + \left(1 - \frac{i}{2} \sum_j \xi_j \right) |x|^2 \right].
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (102) \quad \mathcal{I}_{\varepsilon, \varepsilon'} &= (-i)^{m(m-1)/2} c_m'' \int_{Y'_{2m-q}} \int_{\mathbb{R}^m} \Delta(\lambda) \Delta(\xi) \prod_{j=1}^m |\lambda_j| \\
 &\quad \cdot e^{i\langle \lambda, \xi \rangle} \mathcal{I}(\lambda, \xi) e^{-\varepsilon \sum \xi_j^2 - \varepsilon' \sum \lambda_j^2} d\xi d\lambda,
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{I}(\lambda, \xi) &= \frac{1}{\pi^{dm}} \int_{\mathbb{C}} \int_{\text{Sym}(m, \mathbb{C})} F \left(D(\lambda) + \frac{1}{2} |x|^2 I \right) e^{\Phi(H, x; \xi)} dH dx \\
 &= \frac{1}{\prod_{j \leq k} \left(1 + \frac{i}{2} (\xi_j + \xi_k) \right)} \\
 &\quad \cdot \int_{\mathbb{C}} F \left(D(\lambda) + \frac{1}{2} |x|^2 I \right) e^{-(1 - \frac{i}{2} \sum_j \xi_j) |x|^2} dx.
 \end{aligned}$$

To evaluate the dx integral, we first expand the amplitude:

$$F\left(D(\lambda) + \frac{1}{2}|x|^2 I\right) = F(D(\lambda)) + \left[\frac{4 \sum_{j=1}^m \lambda_j}{m(m+1)(m+3)} - \frac{2}{m+1}\right] |x|^2 + \frac{1}{(m+1)(m+3)} |x|^4,$$

and then integrate to obtain (83).

To evaluate $\lim_{\varepsilon, \varepsilon' \rightarrow 0^+} \mathcal{I}_{\varepsilon, \varepsilon'}$, we first observe as in §4 that the map

$$(\varepsilon_1, \dots, \varepsilon_m) \mapsto \int_{\mathbb{R}^m} \Delta(\xi) e^{i\langle \lambda, \xi \rangle} \mathcal{I}(\lambda, \xi) e^{-\sum \varepsilon_j \xi_j^2} d\xi$$

is a continuous map from $[0, +\infty)^m$ to the tempered distributions. Hence by (98) and (102), we have:

$$(103) \quad \beta_{2q}(m) = \frac{(-i)^{m(m-1)/2}}{4 \pi^{2m} \prod_{j=1}^m j!} \lim_{\varepsilon' \rightarrow 0^+} \lim_{\varepsilon_1, \dots, \varepsilon_m \rightarrow 0^+} m! \int_{Y_{2m-q}} d\lambda \int_{\mathbb{R}^m} d\xi \cdot \Delta(\lambda) \Delta(\xi) \prod_{j=1}^m |\lambda_j| e^{i\langle \lambda, \xi \rangle} \mathcal{I}(\lambda, \xi) e^{-\sum \varepsilon_j \xi_j^2 - \varepsilon' \sum \lambda_j^2}.$$

Letting $\varepsilon_1 \rightarrow 0, \dots, \varepsilon_m \rightarrow 0, \varepsilon' \rightarrow 0$ sequentially, we obtain the formula of Lemma 6.2. q.e.d.

6.4. Values of $\beta_{2q}(m)$. We use the integral formula of Lemma 6.2 to compute the constants $\beta_{2q}(m)$. The ξ_j integrals can be evaluated using residues as in §4.2; the resulting λ integrand is a polynomial function of the λ_j and e^{λ_j} . The integrals were evaluated in dimensions ≤ 4 using Maple 10.¹

In dimension 1, we reproduce the result from [11]:

$$\beta_{21}(1) = \frac{1}{3^3 \cdot \pi}, \quad \beta_{22}(1) = \frac{1}{3^3 \cdot \pi}.$$

In dimension 2, we have:

$$\beta_{22}(2) = \frac{1}{2^3 \cdot 5 \cdot \pi^2}, \quad \beta_{23}(2) = \frac{2^4}{3^4 \cdot 5 \cdot \pi^2}, \quad \beta_{24}(2) = \frac{47}{2^3 \cdot 3^4 \cdot 5 \cdot \pi^2}.$$

In dimension 3, we have:

$$\beta_{23}(3) = \frac{2^2}{5^3 \cdot \pi^3}, \quad \beta_{24}(3) = \frac{11 \cdot 23}{2^5 \cdot 5^3 \cdot \pi^3}, \quad \beta_{25}(3) = \frac{2^9 \cdot 7}{3^6 \cdot 5^3 \cdot \pi^3},$$

$$\beta_{26}(3) = \frac{23563}{2^5 \cdot 3^6 \cdot 5^3 \cdot \pi^3}.$$

¹The Maple program used for this computation (`coefficient.mw`) is included in the source file of the arXiv posting at <http://arxiv.org/e-print/math/0406089>.

In dimension 4, we have:

$$\beta_{24}(4) = \frac{2^2}{3^2 \cdot 7 \cdot \pi^4}, \quad \beta_{25}(4) = \frac{2 \cdot 3 \cdot 41}{5^6 \cdot 7 \cdot \pi^4}, \quad \beta_{26}(4) = \frac{1056667}{2^6 \cdot 3^2 \cdot 5^6 \cdot \pi^4},$$

$$\beta_{27}(4) = \frac{2^{15} \cdot 937}{3^8 \cdot 5^6 \cdot 7 \cdot \pi^4}, \quad \beta_{28}(4) = \frac{267828299}{2^6 \cdot 3^8 \cdot 5^6 \cdot 7 \cdot \pi^4}.$$

This completes the proof of Theorem 1.6 for $m \leq 4$. B. Baugher [2] used Mathematica to compute the constants $\beta_{2q}(5)$, and found that they are positive as well. q.e.d.

Remark. The values of the coefficients $\beta_2(m)$ for the expected total number of critical points are:

$$\beta_2(1) = \frac{2}{3^3 \cdot \pi}, \quad \beta_2(2) = \frac{32}{405\pi^2} = \frac{2^5}{3^4 \cdot 5 \cdot \pi^2},$$

$$\beta_2(3) = \frac{104}{729\pi^3} = \frac{2^3 \cdot 13}{3^6 \cdot \pi^3}, \quad \beta_2(4) = \frac{17152}{45927\pi^4} = \frac{2^8 \cdot 67}{3^8 \cdot 7 \cdot \pi^4}.$$

Appendix 1. Explicit formulas for $\mathbb{C}P^m$

We used Maple 10 to evaluate the integral in Proposition 4.1 to obtain precise formulas² for the expected numbers $\mathcal{N}_{N,q}^{\text{crit}}(\mathbb{C}P^m)$ of critical points of Morse index q of random sections of $H^0(\mathbb{C}P^m, \mathcal{O}(N))$ for $m \leq 6$. In this appendix, we state these formulas in dimensions 1, 2, and 3. These formulas yield the universal leading coefficients $n_q(m)$ of $\mathcal{N}_{N,q,h}^{\text{crit}}$ described in Conjecture 4.4. We give numerical values for these leading coefficients in dimensions $m \leq 6$.

For $m = 1$, we reproduce the result from [11]:

$$\mathcal{N}_{N,1}^{\text{crit}}(\mathbb{C}P^1) = \frac{4(N-1)^2}{3N-2}, \quad \mathcal{N}_{N,2}^{\text{crit}}(\mathbb{C}P^1) = \frac{N^2}{3N-2}; \quad \mathcal{N}_N^{\text{crit}}(\mathbb{C}P^1) = \frac{5N^2-8N+4}{3N-2}.$$

For $m = 2$, we obtain:

$$\mathcal{N}_{N,2}^{\text{crit}}(\mathbb{C}P^2) = \frac{3(N-1)^3}{(2N-1)}, \quad \mathcal{N}_{N,3}^{\text{crit}}(\mathbb{C}P^2) = \frac{16(N-1)^3 N^2}{(3N-2)^3},$$

$$\mathcal{N}_{N,4}^{\text{crit}}(\mathbb{C}P^2) = \frac{N^5(5N-4)}{(3N-2)^3(2N-1)}.$$

Hence, the expected total number of critical points is:

$$\mathcal{N}_N^{\text{crit}}(\mathbb{C}P^2) = \frac{59N^5 - 231N^4 + 375N^3 - 310N^2 + 132N - 24}{(3N-2)^3}.$$

To check the computation, we note that

$$\mathcal{N}_{N,2}^{\text{crit}}(\mathbb{C}P^2) - \mathcal{N}_{N,3}^{\text{crit}}(\mathbb{C}P^2) + \mathcal{N}_{N,4}^{\text{crit}}(\mathbb{C}P^2) = N^2 - 3N + 3 = c_2(T_{\mathbb{C}P^2}^{*1,0} \otimes \mathcal{O}(N)).$$

²The Maple program used for this computation (`exactprojective.mw`) is included in the source file of the arXiv posting at <http://arxiv.org/e-print/math/0406089>.

In the case $m = 3$, we obtain:

$$\begin{aligned}\mathcal{N}_{N,3}^{\text{crit}}(\mathbb{C}\mathbb{P}^3) &= \frac{8(N-1)^4}{(5N-2)}, \\ \mathcal{N}_{N,4}^{\text{crit}}(\mathbb{C}\mathbb{P}^3) &= \frac{(N-1)^4 N^2 (63N^2 - 50N + 10)}{(2N-1)^4 (5N-2)}, \\ \mathcal{N}_{N,5}^{\text{crit}}(\mathbb{C}\mathbb{P}^3) &= \frac{256(N-1)^4 N^5}{(5N-2)(3N-2)^5}, \\ \mathcal{N}_{N,6}^{\text{crit}}(\mathbb{C}\mathbb{P}^3) &= \frac{N^9 (451N^4 - 1248N^3 + 1280N^2 - 576N + 96)}{(2N-1)^4 (3N-2)^5 (5N-2)}.\end{aligned}$$

The expected total number of critical points is:

$$\mathcal{N}_N^{\text{crit}}(\mathbb{C}\mathbb{P}^3) = \frac{637N^8 - 3978N^7 + 11022N^6 - 17608N^5 + 17736N^4 - 11552N^3 + 4768N^2 - 1152N + 128}{(3N-2)^5}.$$

Again, to check the computation:

$$\sum_{q=3}^6 \mathcal{N}_{N,q}^{\text{crit}}(\mathbb{C}\mathbb{P}^3) = N^3 - 4N^2 + 6N - 4 = c_3(T_{\mathbb{C}\mathbb{P}^3}^{*1,0} \otimes \mathcal{O}(N)).$$

By Corollary 1.4, the leading coefficients $n_{m+r}(m) = \frac{\pi^m}{m!} b_{0,q}(m)$ of the N -expansion of $\mathcal{N}_{N,q,h}^{\text{crit}}$ are universal, and hence they are equal to the leading coefficients of $\mathcal{N}_{N,q}^{\text{crit}}(\mathbb{C}\mathbb{P}^m)$. We give in the table below approximate numerical values of these coefficients in dimensions $m \leq 6$:

Leading coefficients $n_{m+r}(m)$ of $\mathcal{N}_{N,m+r}^{\text{crit}}(\mathbb{C}\mathbb{P}^m)$								
r	0	1	2	3	4	5	6	total
$m = 1$	1.33333	0.33333						1.66667
$m = 2$	1.5	0.59259	0.09259					2.18519
$m = 3$	1.6	0.78750	0.21070	0.02320				2.62140
$m = 4$	1.66667	0.93696	0.33019	0.06533	0.00543			3.00457
$m = 5$	1.71429	1.05448	0.44235	0.11939	0.01844	0.00121		3.35015
$m = 6$	1.75	1.14903	0.54457	0.17979	0.03884	0.00486	0.00026	3.66734

Appendix 2. Baugher's conjecture

In this appendix, we describe Baugher's conjectured identity and its implications for the positivity of $\beta_{2q}(m)$. After calculating the integral of each of the individual terms in Lemma 6.2 using Mathematica 5, Baugher made the following conjecture [2]:

Conjecture A.1.

$$\beta_{2q}(m) = \frac{(-i)^{m(m-1)/2}}{4\pi^{2m} \prod_{j=1}^{m-1} j!} \int_{Y_{2m-q}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Delta(\lambda) \Delta(\xi) \prod_{j=1}^m |\lambda_j| \\ \cdot e^{i\langle \lambda, \xi \rangle} \mathcal{J}(\xi) d\xi_1 \cdots d\xi_m d\lambda,$$

where

$$\mathcal{J}(\xi) = \frac{4}{(m+1)(m+2)(m+3) \left(1 - \frac{i}{2} \sum_j \xi_j\right)^2 \prod_{j \leq k} \left[1 + \frac{i}{2}(\xi_j + \xi_k)\right]}.$$

The conjecture has been verified in dimensions $m \leq 5$ [2]. Further, Baugher evaluated this integral by a lengthy calculation:

Proposition A.2 ([2]). *The integral on the right hand side of Conjecture A.1 equals*

$$\frac{4}{(m+1)(m+2)(m+3)} \int_{\mathbf{S}_{m,q-m}} |x|^2 |\det(2HH^* - |x|^2 I)| e^{-\langle (H,x), (H,x) \rangle} dH dx.$$

The expression in Proposition A.2 is obviously positive, and Conjecture A.1 thus implies that $\beta_{2q}(m) > 0$.

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